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Research Article

Admissible Hybrid $\mathcal{Z}$-Contractions in Extended b-Metric Spaces

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In this paper, we define the notation of admissible hybrid $\mathcal{Z}$-contractions in the setting of extended b-metric spaces, which unifies and generalizes previously existing results in literature. Furthermore, as an application, we discuss Ulam-Hyers stability and well-posedness of a fixed point problem.

1. Introduction

The source of metric fixed point theory is considered to be the Banach Contraction Principle which is a very important mechanism for finding the existence and solutions for many problems including differential and integral equations. Afterwards, numerous papers on generalizations and extensions of the Banach's theorem for both single-valued and multivalued mappings have been published, either by changing the contraction conditions or by changing the structure of metric space to more generalized form, e.g., see [1] and all the references therein.

The concept of a b-metric space was accomplished by the works of Bourbaki [2], Bakhtin [3], and Czerwik [4]. Subsequently, several articles have appeared in literature which dealt with the fixed point theorems by taking into account more general forms, of a metric spaces, i.e., b-metric space, see [5], and the applications of relaxed triangular inequalities, like NEM (nonlinear elastic matching), ice floes, etc., were also utilized in various directions [6, 7]. Following the idea of b-metric spaces, a number of authors have presented several results in this direction, see [8, 9]. To have some insight about miscellaneous generalizations of a metric, we refer the readers to [10–15] for some works on b-metric spaces.

In 2017, Kamran et al.[16] generalized the structure of a b-metric space and referred it as an extended b-metric space. He weakened the triangle inequality of a metric and established fixed point results for a class of contractions. Thereafter, many researchers have studied and generalized fixed point results for single and multivalued mappings. Proving extensions of the Banach contraction principle from metric spaces to b-metric spaces and hence to extended b-metric spaces is useful to prove existence and uniqueness theorems for different types of integral and differential equations. Keeping the length of paper concise, we refer to [17–35] and to references mentioned therein. For more topological properties of extended b-metric spaces, see [20].

The main purpose of this paper is to merge different linear and nonlinear results existing in literature in setup of an extended b-metric space, which is a real generalization of a b-metric and a standard metric space. We express our results in a more refined form by combining the notations, like admissible mappings, simulation functions, and hybrid contractions. We will prove fixed point results involving a certain type of mappings. The obtained results generalize [36]. Moreover, we prove Ulam-Hyers stability [37–39] and well-posedness of fixed point problems as well.

2. Preliminaries

In this section, we recollect some definitions and results from literature along with some examples.

Definition 1 [40]. Let $\mathcal{X}$ be a nonempty set and $\theta : \mathcal{X} \times \mathcal{X} \longrightarrow [1,\infty)$. A function $d_\theta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ is called an extended b-metric, if it satisfies the following properties for all $\mu, \gamma, \nu \in \mathcal{X}$:

$(d_\theta 1)$ $d_\theta(\mu, \nu) = 0 \Leftarrow \mu = \nu$

$(d_\theta 2)$ $d_\theta(\mu, \nu) = d_\theta(\nu, \mu)$

$(d_\theta 3)$ $d_\theta(\mu, \gamma) \leq \theta(\mu, \gamma)[d_\theta(\mu, \nu) + d_\theta(\nu, \gamma)]$

The pair $(\mathcal{X}, d_\theta)$ is called an extended b-metric space.

Example 1. Let $\mathcal{X} = [0, 1]$ and $\theta : \mathcal{X} \times \mathcal{X} \longrightarrow [1,\infty)$ defined by $\theta(\mu, \nu) = (1 + \mu + \nu)/(\mu + \nu)$. Define $d_\theta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ as

$d_\theta(\mu, \nu) = 1/\mu\nu$ for $\mu, \nu \in (0, 1], \mu \neq \nu$

$d_\theta(\mu, \nu) = 0$ for $\mu, \nu \in [0, 1], \mu = \nu$

$d_\theta(\mu, 0) = d_\theta(0, \mu) = 1/\mu\nu$ for $\mu \in (0, 1]$

Note that $(\mathcal{X}, d_\theta)$ is an extended b-metric space.

Example 2. Let $\mathcal{X} = \{1, 2, 3\}$, $\theta : \mathcal{X} \times \mathcal{X} \longrightarrow [1,\infty)$, and $d_\theta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ as

$\theta(\mu, \nu) = 1 + \mu + \nu$ such that

$d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0$

$d_\theta(1, 2) = d_\theta(2, 1) = 70$

$d_\theta(1, 3) = d_\theta(3, 1) = 90$

$d_\theta(2, 3) = d_\theta(3, 2) = 20$

Here, d_θ is an extended b-metric on $\mathcal{X}$.

Note that the extended b-metric space becomes a b-metric space, whenever $\theta(\mu, \nu) = \delta$, where $\delta \geq 1$ and a standard metric space for $\delta = 1$.

Definition 2 [16]. Let $(\mathcal{X}, d_\theta)$ be an extended b-metric space. The sequence $\{\mu_n\}$ in $\mathcal{X}$ is termed as follows:

(i) Cauchy if and only if $d_\theta(\mu_n, \mu_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$

(ii) Convergent if and only if there exists $\mu \in \mathcal{X}$ such that $d_\theta(\mu_n, \mu) \longrightarrow 0$ as $n \longrightarrow \infty$ and we write $\lim_{n\longrightarrow\infty}\mu_n = \mu$

Note that the extended b-metric space $(\mathcal{X}, d_\theta)$ is complete if every Cauchy sequence is convergent.

The b-metric is not continuous in general and so the same for an extended b-metric. We define the concept of f-orbital continuity (in case of an extended b-metric space) as used in [41].

Definition 3 [42]. Given a mapping $f : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{X}$. Suppose that there exists some $\mu_0 \in \mathcal{D}$ such that $\mathcal{O}(\mu_0) = \{\mu_0, f\mu_0, f^2\mu_0, \cdots\} \subset \mathcal{D}$. The set $\mathcal{O}(\mu_0)$ is called the orbit of $\mu_0 \in \mathcal{D}$. A self-mapping $f : \mathcal{X} \longrightarrow \mathcal{X}$ is called orbitally continuous if $\lim_{n\longrightarrow\infty} f^n(\eta) = \eta$ for some $\eta \in \mathcal{X}$ implies that

$$\lim_{n\longrightarrow\infty} f(f^n(\eta)) = f(\eta). \tag{1}$$

Moreover, if every Cauchy sequence of the form $\{f^n(\eta)\}$ as $n \longrightarrow \infty$, $\eta \in \mathcal{X}$ converges in $(\mathcal{X}, d_\theta)$, then an extended b-metric space $(\mathcal{X}, d_\theta)$ is called f-orbitally complete.

Definition 4 [40]. Let $(\mathcal{X}, d_\theta)$ be an extended b-metric space. A function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called an extended b-comparison function if it is increasing and also there exists a mapping $f : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{X}$ such that for some $\mu_0 \in \mathcal{D}$, $\mathcal{O}(\mu_0) \subset \mathcal{D}$, $\sum_{n=0}^{\infty}\phi^n(v)\prod_{i=0}^{n}\theta(\mu_i, \mu_m)$ converges for each $v \in \mathbb{R}^+$ and for every $m \in \mathbb{N}$. Here, $\mu_n = f^n\mu_0$ for $n = 0, 1, 2, \cdots$. We say that ϕ is an extended b-comparison function for f at μ_0 and denotes the collection of all extended b-comparison functions by Ψ_s.

Example 3 [40]. Let $(\mathcal{X}, d_\theta)$ be an extended b-metric space and f be a self mapping on $\mathcal{X}$. Assume that $\lim_{n,m\longrightarrow\infty}\theta(\mu_n, \mu_m)$ exists. Define $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(v) = \lambda v$, with $\lim_{n,m\longrightarrow\infty}\theta(\mu_n, \mu_m) < 1/\lambda$. Then, the series, $\sum_{n=0}^{\infty}\phi^n(v)\prod_{i=0}^{n}\theta(\mu_i, \mu_m)$ converges by ratio test.

Here, $\lambda \in [0, 1)$ and $\mu_n = f^n\mu_0$ for $n = 1, 2, \cdots$.

The notation of α-admissible mappings also played a vital role in fixed point theory, see [43, 44].

Definition 5 [36]. Let $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ be a mapping. A function $f : \mathcal{X} \longrightarrow \mathcal{X}$ is α-orbital admissible if $\alpha(\mu, f\mu) \geq 1\alpha(f\mu, f^2\mu) \geq 1$.

An α-orbital admissible, mapping f is called triangular α-orbital admissible, if $\alpha(\mu, \nu) \geq 1$ and $\alpha(\nu, f\nu) \geq 1\alpha(\mu, f\nu) \geq 1$ for all $\mu, \nu \in \mathcal{X}$.

Example 4. Let $\mathcal{X} = \{0, 1, 2, 3\}$ and $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that $f0 = 0$ and $f1 = f2 = f3 = 1$. Consider $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ given as $\alpha(1, 2) = \alpha(2, 1) = \alpha(1, 3) = \alpha(3, 1) = \alpha(1, 1) = 1$ and 0 otherwise. Clearly, f is α-orbital admissible.

Definition 6 [45]. A mapping $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ satisfying the following conditions

$(\zeta 1)$ $\zeta(t, s) < s - t$ for all $t, s > 0$

$(\zeta 2)$ If $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are the sequences in $(0, \infty)$ such that

$\lim_{n\longrightarrow\infty}t_n = \lim_{n\longrightarrow\infty}s_n > 0$. Then, $\limsup_{n\longrightarrow\infty}\zeta(t_n, s_n) < 0$ is termed as a simulation function

Let $\mathcal{Z}$ represents the set of all simulation functions defined above.

The main idea of the simulation function is very useful and effective. For a self-mapping $\mathcal{T}$ on a metric space, the contraction $d(\mathcal{T}\mu, \mathcal{T}\nu) \leq \kappa d(\mu, \nu)$ can be represented as $0 \leq \kappa d(\mu, \nu) - d(\mathcal{T}\mu, \mathcal{T}\nu) = \zeta(d(\mu, \nu), d(\mathcal{T}\mu, \mathcal{T}\nu))$, where $\kappa \in [0, 1)$ and $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$. By letting $d(\mu, \nu) = s$ and $d(\mathcal{T}\mu, \mathcal{T}\nu) = t$, the corresponding simulation function for Banach's fixed point theorem is $\zeta(t, s) = \kappa s - t$. It is clear that for many other well-known results (Geraghty, Boyd-Wong, etc.), one can find a corresponding simulation function, see [36, 46–51]. In other words, a simulation function can be considered as a generator of different contraction type inequalities.

Definition 7. A self-mapping f, defined on a metric space $(\mathcal{X}, d)$, is called a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, if it satisfies

$$\zeta(d(f\mu, f\nu), d(\mu, \nu)) \geq 0 \text{ for all } \mu, \nu \in \mathcal{X}. \tag{2}$$

Every $\mathcal{Z}$-contraction defined on a complete metric space has a 0 fixed point, as described in [46]. A $\mathcal{Z}$-contraction generalizes the Banach contraction principle by assuming $\gamma \in [0, 1)$ and $\zeta(t, s) = \gamma s - t$ for all $s, t \in [0,\infty)$. It also unifies several known type of contractions. Many authors have extended their work on $\mathcal{Z}$-contractions in order to prove a more generalized version (see [47]).

The notion of admissible hybrid contractions is introduced [36, 46, 48–50] in order to generalize and unify the several existing fixed point results in the literature. The main goal of this paper is to investigate the existence and uniqueness of a fixed point of admissible hybrid $\mathcal{Z}$-contractions in the context of an extended b-metric space. We shall also list some existing results in the literature as corollaries and consequences of our main results. Consequently, the results in the class of b-metric spaces and standard metric spaces become a special case of our obtained results.

3. Main Results

Definition 8. Let $(\mathcal{X}, d_\theta)$ be an extended b-metric space. A self-mapping f is called an admissible hybrid contraction if there exist an extended b-comparison function $\psi : [0,\infty) \longrightarrow [0,\infty) \in \Psi_s$ and $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ such that

$$\alpha(\mu, \nu) d_\theta(f\mu, f\nu) \leq \psi\left(\mathcal{P}_f^r(\mu, \nu)\right), \tag{3}$$

where $r \geq 0$ and $\lambda_i \geq 0$, $i = 1, 2, 3, 4, 5$ with $\sum_{i=1}^{5} \lambda_i = 1$, and

$$\mathcal{P}_f^r(\mu, \nu) = \begin{cases} [\mathcal{Q}(\mu, \nu)]^{1/r}, & \text{for } r > 0 \text{ and } \mu, \nu \in \mathcal{X}, \\ \mathcal{R}(\mu, \nu), & \text{for } r = 0 \text{ and } \mu, \nu \in \mathcal{X}, \end{cases} \tag{4}$$

where

$$\begin{aligned} \mathcal{Q}(\mu, \nu) := {} & \lambda_1 d_\theta^r(\mu, \nu) + \lambda_2 d_\theta^r(\mu, f\mu) + \lambda_3 d_\theta^r(\nu, f\nu) + \lambda_4 \\ & \cdot \left(\frac{d_\theta(\nu, f\nu)(1 + d_\theta(\mu, f\mu))}{1 + d_\theta(\mu, \nu)}\right)^r \\ & + \lambda_5 \left(\frac{d_\theta(\nu, f\mu)(1 + d_\theta(\mu, f\nu))}{1 + d_\theta(\mu, \nu)}\right)^r, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \mathcal{R}(\mu, \nu) := {} & d_\theta^{\lambda_1}(\mu, \nu) \cdot d_\theta^{\lambda_2}(\mu, f\mu) \cdot d_\theta^{\lambda_3}(\nu, f\nu) \\ & \cdot \left(\frac{d_\theta(\nu, f\nu)(1 + d_\theta(\mu, f\mu))}{1 + d_\theta(\mu, \nu)}\right)^{\lambda_4} \\ & \cdot \left(\frac{d_\theta(\mu, f\nu) + d_\theta(\nu, f\mu)}{2\theta(\mu, f\nu)}\right)^{\lambda_5}. \end{aligned} \tag{6}$$

Definition 9. Let $(\mathcal{X}, d_\theta)$ be an extended b-metric space. A self-mapping f is said to be an admissible hybrid $\mathcal{Z}$-contraction, if there exist an extended b-comparison function $\psi \in \Psi_s$, $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$, and $\zeta \in \mathcal{Z}$ such that

$$\zeta\left(\alpha(\mu, \nu) d_\theta(f\mu, f\nu), \psi\left(\mathcal{P}_f^r(\mu, \nu)\right)\right) \geq 0, \forall \mu, \nu \in \mathcal{X}. \tag{7}$$

Further, we discuss the existence and uniqueness of a fixed point of an admissible hybrid $\mathcal{Z}$-contraction mapping.

Note that we assume that d_θ is continuous and $\lim_{m,n\longrightarrow\infty} \theta(\mu_m, \mu_n) < \infty$, throughout Section 3 and Section 4.

Theorem 10. *Let $(\mathcal{X}, d_\theta)$ be an extended b-metric space. Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an admissible hybrid $\mathcal{Z}$-contraction, which satisfies the following axioms:*

(i) The function f is triangular α-orbital admissible

(ii) There exists $\mu_0 \in \mathcal{X}$ such that $\alpha(\mu_0, f\mu_0) \geq 1$

(iii) Either f is continuous or

(iv) f^2 is continuous and $\alpha(\mu, f\mu) \geq 1$ for any $\mu \in Fix_{f^2}(\mathcal{X})$

Then, f possesses a fixed point.

Proof. Let $\mu_0 \in \mathcal{X}$ be any arbitrary point. We start from μ_0 and iteratively, we construct a sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $\mu_n = f^n \mu_0$ for all $n \in \mathbb{N}$. Suppose that there exists some $m \in \mathbb{N}$ such that $f\mu_m = \mu_{m+1} = \mu_m$, we find that μ_m is a fixed point of f, and in this way, the proof is completed. Thus, we can assume that $\mu_n \neq \mu_{n-1}$ for any $n \in \mathbb{N}$. By condition (i), as f is an admissible hybrid $\mathcal{Z}$-contraction, so by assuming $\mu = \mu_{n-1}$ and $\nu = \mu_n$ in equation (3), we have

$$\begin{aligned} 0 & \leq \zeta\left(\alpha(\mu_{n-1}, \mu_n) d_\theta(f\mu_{n-1}, f\mu_n), \psi\left(\mathcal{P}_f^r(\mu_{n-1}, \mu_n)\right)\right) \\ & < \psi\left(\mathcal{P}_f^r(\mu_{n-1}, \mu_n) - (\alpha(\mu_{n-1}, \mu_n) d_\theta(f\mu_{n-1}, f\mu_n)\right), \end{aligned} \tag{8}$$

which gives,

$$\alpha(\mu_{n-1},\mu_n)d_\theta(f\mu_{n-1},f\mu_n) \le \psi\left(\mathcal{P}_f^r(\mu_{n-1},\mu_n)\right). \quad (9)$$

By condition (ii), as f is triangular α-orbital admissible and $\alpha(\mu_{n-1},\mu_n) \ge 1$, so

$$d_\theta(\mu_n,\mu_{n+1}) \le \alpha(\mu_{n-1},\mu_n)d_\theta(f\mu_{n-1},f\mu_n) \le \psi\left(\mathcal{P}_f^r(\mu_{n-1},\mu_n)\right). \quad (10)$$

□

Case 1. Consider $r > 0$ so that

$$\begin{aligned}
&\mathcal{P}_f^r(\mu_{n-1},\mu_n)\\
&= \Big[\lambda_1 d_\theta^r(\mu_{n-1},\mu_n) + \lambda_2 d_\theta^r(\mu_{n-1},f\mu_{n-1})\\
&\quad + \lambda_3 d_\theta^r(\mu_n,f\mu_n)\\
&\quad + \lambda_4\left(\frac{d_\theta(\mu_n,f\mu_n)(1+d_\theta(\mu_{n-1},f\mu_{n-1}))}{1+d_\theta(\mu_{n-1},\mu_n)}\right)^r\\
&\quad + \lambda_5\left(\frac{d_\theta(\mu_n,f\mu_{n-1})(1+d_\theta(\mu_{n-1},f\mu_n))}{1+d_\theta(\mu_{n-1},\mu_n)}\right)^r\Big]^{1/r}\\
&= \Big[\lambda_1 d_\theta^r(\mu_{n-1},\mu_n) + \lambda_2 d_\theta^r(\mu_{n-1},\mu_n) + \lambda_3 d_\theta^r(\mu_n,\mu_{n+1})\\
&\quad + \lambda_4\left(\frac{d_\theta(\mu_n,\mu_{n+1})(1+d_\theta(\mu_{n-1},\mu_n))}{1+d_\theta(\mu_{n-1},\mu_n)}\right)^r\\
&\quad + \lambda_5\left(\frac{d_\theta(\mu_n,\mu_n)(1+d_\theta(\mu_{n-1},\mu_{n+1}))}{1+d_\theta(\mu_{n-1},\mu_n)}\right)^r\Big]^{\lambda_5\left(\frac{d_\theta(\mu_n,\mu_n)(1+d_\theta(\mu_{n-1},\mu_{n+1}))}{1+d_\theta(\mu_{n-1},\mu_n)}\right)}\\
&= [\lambda_1 d_\theta^r(\mu_{n-1},\mu_n) + \lambda_2 d_\theta^r(\mu_{n-1},\mu_n)\\
&\quad + \lambda_3 d_\theta^r(\mu_n,\mu_{n+1}) + \lambda_4(d_\theta(\mu_n,\mu_{n+1}))^r]^{1/r}\\
&= [(\lambda_1+\lambda_2)d_\theta^r(\mu_{n-1},\mu_n) + \lambda_3 + (\lambda_4)d_\theta^r(\mu_n,\mu_{n+1})]^{1/r}.
\end{aligned} \quad (11)$$

From equation (10),

$$\begin{aligned}
d_\theta(\mu_n,\mu_{n+1}) &\le \alpha(\mu_{n-1},\mu_n)d_\theta(f\mu_{n-1},f\mu_n) \le \psi\left(\mathcal{P}_f^r(\mu_{n-1},\mu_n)\right)\\
&= \psi\left([(\lambda_1+\lambda_2)d_\theta^r(\mu_{n-1},\mu_n) + (\lambda_3+\lambda_4)d_\theta^r(\mu_n,\mu_{n+1})]^{1/r}\right).
\end{aligned} \quad (12)$$

Suppose $d_\theta(\mu_{n-1},\mu_n) \le d_\theta(\mu_n,\mu_{n+1})$. As ψ is an increasing function, so above inequality can be written as

$$\begin{aligned}
d_\theta(\mu_{n+1},\mu_n) &\le \alpha(\mu_{n-1},\mu_n)d_\theta(f\mu_{n-1},f\mu_n)\\
&\le \psi\left([(\lambda_1+\lambda_2)d_\theta^r(\mu_{n-1},\mu_n) + (\lambda_3+\lambda_4)d_\theta^r(\mu_n,\mu_{n+1})]^{1/r}\right)\\
&\le \psi\left([(\lambda_1+\lambda_2+\lambda_3+\lambda_4)d_\theta^r(\mu_n,\mu_{n+1})]^{1/r}\right)\\
&= \psi\left([(\lambda_1+\lambda_2+\lambda_3+\lambda_4)]^{1/r}d_\theta(\mu_n,\mu_{n+1})\right).
\end{aligned} \quad (13)$$

As $\psi(t) < t$, so

$$d_\theta(\mu_{n+1},\mu_n) < [(\lambda_1+\lambda_2+\lambda_3+\lambda_4)]^{1/r}d_\theta(\mu_n,\mu_{n+1}). \quad (14)$$

Since $\lambda_1+\lambda_2+\lambda_3+\lambda_4 \le 1$, we get

$$d_\theta(\mu_{n+1},\mu_n) < [(\lambda_1+\lambda_2+\lambda_3+\lambda_4)]^{1/r}d_\theta(\mu_n,\mu_{n+1}) \le d_\theta(\mu_n,\mu_{n+1}), \quad (15)$$

which is a contradiction. Thus, for every $n \in \mathbb{N}$, $d_\theta(\mu_n,\mu_{n+1}) < d_\theta(\mu_{n-1},\mu_n)$, and thus equation (10) becomes

$$\begin{aligned}
d_\theta(\mu_n,\mu_{n+1}) &\le \psi\left([(\lambda_1+\lambda_2)d_\theta^r(\mu_{n-1},\mu_n) + (\lambda_3+\lambda_4)d_\theta^r(\mu_n,\mu_{n+1})]^{1/r}\right)\\
&< \psi\left([(\lambda_1+\lambda_2+\lambda_3+\lambda_4)]^{1/r}d_\theta(\mu_{n-1},\mu_n)\right)\\
&\le \psi(d_\theta(\mu_{n-1},\mu_n))\psi^2(d_\theta(\mu_{n-2},\mu_{n-1}))\\
&\le \cdots \le \psi^n(d_\theta(\mu_0,\mu_1)).
\end{aligned} \quad (16)$$

Let $m > n$, by triangular inequality, we have

$$\begin{aligned}
d_\theta(\mu_n,\mu_m) &\le \theta(\mu_n,\mu_m)d_\theta(\mu_n,\mu_{n+1})\\
&\quad + \theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m)d_\theta(\mu_{n+1},\mu_{n+2}) + \cdots + \theta\\
&\quad \cdot (\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m)\theta(\mu_{n+2},\mu_m)\cdots\theta\\
&\quad \cdot (\mu_{m-2},\mu_m)\theta(\mu_{m-1},\mu_m)d_\theta(\mu_{m-1},\mu_m)\\
&\le \theta(\mu_n,\mu_m)\psi^n(d_\theta(\mu_0,\mu_1)) + \theta(\mu_n,\mu_m)\theta\\
&\quad \cdot (\mu_{n+1},\mu_m)\psi^{n+1}(d_\theta(\mu_0,\mu_1)) + \cdots + \theta(\mu_n,\mu_m)\theta\\
&\quad \cdot (\mu_{n+1},\mu_m)\theta(\mu_{n+2},\mu_m)\cdots\theta(\mu_{m-2},\mu_m)\theta\\
&\quad \cdot (\mu_{m-1},\mu_m)\psi^{m-1}(d_\theta(\mu_0,\mu_1))\\
&\le \theta(\mu_1,\mu_m)\theta(\mu_2,\mu_m)\cdots\theta(\mu_{n-1},\mu_m)\theta(\mu_n,\mu_m)\psi^n\\
&\quad \cdot (d_\theta(\mu_0,\mu_1)) + \theta(\mu_1,\mu_m)\theta(\mu_2,\mu_m)\cdots\theta\\
&\quad \cdot (\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m)\psi^{n+1}(d_\theta(\mu_0,\mu_1)) + \cdots + \theta\\
&\quad \cdot (\mu_1,\mu_m)\theta(\mu_2,\mu_m)\cdots\theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m)\cdots\theta\\
&\quad \cdot (\mu_{m-2},\mu_m)\theta(\mu_{m-1},\mu_m)\psi^{m-1}(d_\theta(\mu_0,\mu_1)).
\end{aligned} \quad (17)$$

Since ψ is an extended b-comparison function, the series

$$\sum_{n=1}^{\infty}\psi^n d_\theta(\mu_0,\mu_1)\prod_{x=1}^{n}\theta(\mu_x,\mu_m), \quad (18)$$

is convergent for every $m \in \mathbb{N}$.

Denote $\mathcal{S} = \sum_{n=1}^{\infty}\psi^n d_\theta(\mu_0,\mu_1)\prod_{x=1}^{n}\theta(\mu_x,\mu_m)$ and $\mathcal{S}_n = \sum_{j=1}^{n}\psi^j d_\theta(\mu_0,\mu_1)\prod_{x=1}^{j}\theta(\mu_x,\mu_m)$.

Thus, for $m > n$, the above inequality becomes

$$d_\theta(\mu_n,\mu_m) \le [\mathcal{S}_{m-1} - \mathcal{S}_n]. \quad (19)$$

Letting $n, m \longrightarrow \infty$, we get

$$d_\theta(\mu_n,\mu_m) \longrightarrow 0, \quad (20)$$

which implies that $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete extended b-metric space. Therefore, it is convergent, so there exists $\mu' \in \mathbb{N}$ such that

$$\lim_{n\longrightarrow\infty} d_\theta\left(\mu_n, \mu'\right) = 0. \tag{21}$$

Now, we prove that μ' is a fixed point of f. By condition (iii), if f is continuous, we have

$$\begin{aligned} d_\theta\left(\mu', f\mu'\right) &= \lim_{n\longrightarrow\infty} d_\theta(\mu_n, f\mu_n) = \lim_{n\longrightarrow\infty} d_\theta(\mu_n, \mu_{n+1}) = 0, \\ \mu' &= f\mu'. \end{aligned} \tag{22}$$

Therefore, μ' is a fixed point of f. Now, consider that f^2 is continuous. It follows that $f^2\mu' = \lim_{n\longrightarrow\infty} f^2\mu_n = \mu'$. We shall now prove that $f\mu' = \mu'$. Contrarily, suppose that, $f\mu' \neq \mu'$, from equation (3)

$$\begin{aligned} 0 &\leq \zeta\left(\alpha\left(f\mu', \mu'\right) d_\theta\left(f^2\mu', f\mu'\right), \psi\left(\mathscr{P}_f^r\left(f\mu', \mu'\right)\right)\right) \\ &= \psi\left(\mathscr{P}_f^r\left(f\mu', \mu'\right)\right) - \alpha\left(f\mu', \mu'\right) d_\theta\left(f^2\mu', f\mu'\right). \end{aligned} \tag{23}$$

It implies that

$$d_\theta\left(\mu', f\mu'\right) = d_\theta\left(f^2\mu', f\mu'\right) \leq \alpha\left(f\mu', \mu'\right) d_\theta\left(f\mu', \mu'\right). \tag{24}$$

As $\psi(t) < t$, so

$$\begin{aligned} &\psi\left(\mathscr{P}_f^r\left(f\mu', \mu'\right)\right) < \mathscr{P}_f^r\left(f\mu', \mu'\right), \\ \mathscr{P}_f^r\left(f\mu', \mu'\right) &= \left[\lambda_1 d_\theta^r\left(f\mu', \mu'\right) + \lambda_2 d_\theta^r\left(\mu', f\mu'\right) + \lambda_3 d_\theta^r\left(f\mu', f^2\mu'\right)\right. \\ &\quad + \lambda_4\left(\frac{d_\theta\left(\mu', f\mu'\right)\left(1 + d_\theta\left(f\mu', f^2\mu'\right)\right)}{1 + d_\theta\left(\mu', f\mu'\right)}\right)^r \\ &\quad \left. + \lambda_5\left(\frac{d_\theta\left(f\mu', f\mu'\right)\left(1 + d_\theta\left(\mu', f^2\mu'\right)\right)}{1 + d_\theta\left(\mu', f\mu'\right)}\right)^r\right]^{1/r} \\ &= \left[\lambda_1 d_\theta^r\left(f\mu', \mu'\right) + \lambda_2 d_\theta^r\left(\mu', f\mu'\right) + \lambda_3 d_\theta^r\left(f\mu', \mu'\right)\right. \\ &\quad + \lambda_4\left(\frac{d_\theta\left(\mu', f\mu'\right)\left(1 + d_\theta\left(f\mu', \mu'\right)\right)}{1 + d_\theta\left(\mu', f\mu'\right)}\right)^r \\ &\quad \left. + \lambda_5\left(\frac{d_\theta\left(f\mu', f\mu'\right)\left(1 + d_\theta\left(\mu', \mu'\right)\right)}{1 + d_\theta\left(\mu', f\mu'\right)}\right)^r\right]^{1/r} \\ &\leq \left[(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d_\theta\left(\mu', f\mu'\right)\right]^{1/r} \\ &= [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)]^{1/r} d_\theta^r\left(\mu', f\mu'\right) \leq d_\theta\left(\mu', f\mu'\right), \end{aligned} \tag{25}$$

which leads to a contradiction. Thus, $f\mu' = \mu'$.

Case 2. Consider $r = 0$. Let $\mu = \mu_{n-1}$ and $\nu = \mu_n$. One writes

$$\begin{aligned} \mathscr{P}_f^r(\mu_{n-1}, \mu_n) &= d_\theta^{\lambda_1}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_2}(\mu_{n-1}, f\mu_{n-1}) \cdot d_\theta^{\lambda_3}(\mu_n, f\mu_n) \\ &\quad \cdot \left(\frac{d_\theta(\mu_n, f\mu_n)(1 + d_\theta(\mu_{n-1}, f\mu_{n-1}))}{1 + d_\theta(\mu_{n-1}, \mu_n)}\right)^{\lambda_4} \\ &\quad \cdot \left(\frac{d_\theta(\mu_{n-1}, f\mu_n) + d_\theta(\mu_n, f\mu_{n-1})}{2\theta(\mu_{n-1}, f\mu_n)}\right)^{\lambda_5} \\ &= d_\theta^{\lambda_1}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_2}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_3}(\mu_n, \mu_{n+1}) \\ &\quad \cdot \left(\frac{d_\theta(\mu_n, \mu_{n+1})(1 + d_\theta(\mu_{n-1}, \mu_n))}{1 + d_\theta(\mu_{n-1}, \mu_n)}\right)^{\lambda_4} \\ &\quad \cdot \left(\frac{d_\theta(\mu_{n-1}, \mu_{n+1}) + d_\theta(\mu_n, \mu_n)}{2\theta(\mu_{n-1}, \mu_{n+1})}\right)^{\lambda_5} \\ &= d_\theta^{\lambda_1}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_2}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_3}(\mu_n, \mu_{n+1}) \\ &\quad \cdot d_\theta^{\lambda_4}(\mu_n, \mu_{n+1}) \cdot \left(\frac{d_\theta(\mu_{n-1}, \mu_{n+1})}{2\theta(\mu_{n-1}, \mu_{n+1})}\right)^{\lambda_5}. \end{aligned} \tag{26}$$

Using triangular inequality, we obtain

$$\begin{aligned} \mathscr{P}_f^r(\mu_{n-1}, \mu_n) &\leq d_\theta^{\lambda_1}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_2}(\mu_{n-1}, \mu_n) \\ &\quad \cdot d_\theta^{\lambda_3}(\mu_n, \mu_{n+1}) \cdot d_\theta^{\lambda_4}(\mu_n, \mu_{n+1}) \\ &\quad \cdot \left(\frac{\theta(\mu_{n-1}, \mu_{n+1})[d_\theta(\mu_{n-1}, \mu_n) + d_\theta(\mu_n, \mu_{n+1})]}{2\theta(\mu_{n-1}, \mu_{n+1})}\right)^{\lambda_5} \\ &\leq d_\theta^{\lambda_1}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_2}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_3}(\mu_n, \mu_{n+1}) \\ &\quad \cdot d_\theta^{\lambda_4}(\mu_n, \mu_{n+1}) \cdot \left(\frac{d_\theta(\mu_{n-1}, \mu_n) + d_\theta(\mu_n, \mu_{n+1})}{2}\right)^{\lambda_5}. \end{aligned} \tag{27}$$

By using the following inequality, one gets

$$\left(\frac{x + y}{2}\right)^c \leq \frac{x^c + y^c}{2}, \forall x, y, c > 0. \tag{28}$$

So, we have

$$\begin{aligned} \mathscr{P}_f^r(\mu_{n-1}, \mu_n) &\leq d_\theta^{\lambda_1}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_2}(\mu_{n-1}, \mu_n) \cdot d_\theta^{\lambda_3}(\mu_n, \mu_{n+1}) \\ &\quad \cdot d_\theta^{\lambda_4}(\mu_n, \mu_{n+1}) \cdot \frac{d_\theta(\mu_{n-1}, \mu_n)^{\lambda_5} + d_\theta(\mu_n, \mu_{n+1})^{\lambda_5}}{2}, \end{aligned} \tag{29}$$

and by equation (7),

$$\begin{aligned} 0 &\leq \zeta\left(\alpha(\mu_{n-1}, \mu_n) d_\theta(f\mu_{n-1}, f\mu_n), \psi\left(\mathscr{P}_f^r(\mu_{n-1}, \mu_n)\right)\right) \\ &< \psi\left(\mathscr{P}_f^r(\mu_{n-1}, \mu_n) - \alpha(\mu_{n-1}, \mu_n) d_\theta(f\mu_{n-1}, f\mu_n)\right), \end{aligned} \tag{30}$$

which gives,

$$d_\theta(\mu_n, \mu_{n+1}) \le \alpha(\mu_{n-1}, \mu_n) d_\theta(f\mu_{n-1}, f\mu_n) \le \psi\left(\mathscr{P}_f^r(\mu_{n-1}, \mu_n)\right). \tag{31}$$

Suppose that $d_\theta(\mu_{n-1}, \mu_n) \le d_\theta(\mu_n, \mu_{n+1})$. As ψ is an increasing function, so

$$\begin{aligned} &d_\theta(\mu_n, \mu_{n+1}) \le d_\theta^{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5}(\mu_n, \mu_{n+1}) = d_\theta(\mu_n, \mu_{n+1}), \\ &d_\theta(\mu_n, \mu_{n+1}) \le d_\theta(\mu_n, \mu_{n+1}), \end{aligned} \tag{32}$$

so, we have two cases here that $d_\theta(\mu_n, \mu_{n+1}) = d_\theta(\mu_n, \mu_{n+1})$ or $d_\theta(\mu_n, \mu_{n+1}) < d_\theta(\mu_n, \mu_{n+1})$ which is a contradiction. Thus, we obtain

$$d_\theta(\mu_n, \mu_{n+1}) \le \psi\left(\mathscr{P}_f^r(\mu_{n-1}, \mu_n)\right) < \psi^n d_\theta(\mu_0, \mu_1). \tag{33}$$

Utilizing the same arguments as of Case 1, we obtain that $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in an extended b-metric space. Therefore, it is convergent, so there exists $\mu' \in \mathbb{N}$ such that

$$\lim_{n\longrightarrow\infty} \mu_n = \mu'. \tag{34}$$

Now, further we claim that μ' is a fixed point of f. By condition (iii), if f is continuous, then

$$d_\theta\left(\mu', f\mu'\right) = \lim_{n\longrightarrow\infty} d_\theta(\mu_n, f\mu_n) = \lim_{n\longrightarrow\infty} d_\theta(\mu_n, \mu_{n+1}) = 0. \tag{35}$$

Thus,

$$\mu' = f\mu'. \tag{36}$$

Hence, μ' is a fixed point of f. Now, consider f^2 is continuous. It follows that $f^2\mu' = \mu'$. Now, we shall validate that $f\mu' = \mu'$. Contrarily, suppose that $f\mu' \ne \mu'$, then

$$\begin{aligned} 0 &\le \zeta\left(\alpha\left(f\mu', \mu'\right) d_\theta\left(f^2\mu', f\mu'\right), \psi\left(\mathscr{P}_f^r\left(f^2\mu', f\mu'\right)\right)\right) \\ &= \psi\left(\mathscr{P}_f^r\left(f^2\mu', f\mu'\right)\right) - \alpha\left(f\mu', \mu'\right) d_\theta\left(f^2\mu', f\mu'\right). \end{aligned} \tag{37}$$

It implies that

$$\begin{aligned} d_\theta\left(\mu', f\mu'\right) &= d_\theta\left(f^2\mu', f\mu'\right) \le \alpha\left(f\mu', \mu'\right) d_\theta\left(f^2\mu', f\mu'\right) \\ &\le \psi\left(\mathscr{P}_f^r\left(f^2\mu', f\mu'\right)\right) = \psi\left(\mathscr{P}_f^r\left(\mu', f\mu'\right)\right), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \mathscr{P}_f^r\left(\mu', f\mu'\right) &= d_\theta^{\lambda_1+\lambda_2+\lambda_3}\left(\mu', f\mu'\right) \\ &\cdot \left(\frac{d_\theta\left(\mu', f\mu'\right)\left(1 + d_\theta\left(\mu', f\mu'\right)\right)}{1 + d_\theta\left(\mu', f\mu'\right)}\right)^{\lambda_4} \\ &\cdot \left(\frac{d_\theta\left(\mu', f^2\mu'\right) + d_\theta\left(f\mu', f\mu'\right)}{2\theta\left(\mu', f\mu'\right)}\right)^{\lambda_5} \\ &= d_\theta^{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\left(\mu', f\mu'\right) < d_\theta\left(\mu', f\mu'\right). \end{aligned} \tag{39}$$

Therefore, we have

$$d_\theta\left(\mu', f\mu'\right) \le \psi\left(\mathscr{P}_f^r\left(\mu', f\mu'\right)\right) < \psi\left(d_\theta\left(\mu', f\mu'\right)\right) < d_\theta\left(\mu', f\mu'\right), \tag{40}$$

which is a contradiction. So, $f\mu' = \mu'$.

Theorem 11. *Assume that all the postulates of Theorem 10 hold. Additionally, suppose that $\alpha(\mu', \nu') \ge 1$, for any $\mu', \nu' \in Fix_f(\mathscr{X})$. Then, f has a unique fixed point.*

Proof. Assume that the fixed point of the mapping f is not unique. Let $\nu' \in \mathscr{X}$ be another fixed point of f, where $\mu' \ne \nu'$. For case $r > 0$, by equation (7), we get

$$\begin{aligned} 0 &\le \zeta\left(\alpha\left(\mu', \nu'\right) d_\theta\left(f\mu', f\nu'\right), \psi\left(\mathscr{P}_f^r\left(\mu', \nu'\right)\right)\right) \\ &< \psi\left(\mathscr{P}_f^r\left(\mu', \nu'\right)\right) - \alpha\left(\mu', \nu'\right) d_\theta\left(f\mu', f\nu'\right). \end{aligned} \tag{41}$$

This yields that

$$\begin{aligned} d_\theta\left(\mu', \nu'\right) &\le \alpha\left(\mu', \nu'\right) d_\theta\left(f\mu', f\nu'\right) \\ &\le \psi\left(\mathscr{P}_f^r\left(\mu', \nu'\right)\right) < \mathscr{P}_f^r\left(\mu', \nu'\right) \\ &= \Bigg[\lambda_1 d_\theta^r\left(\mu', \nu'\right) + \lambda_2 d_\theta^r\left(\mu', f\mu'\right) + \lambda_3 d_\theta^r\left(\nu', f\nu'\right) \\ &\quad + \lambda_4\left(\frac{d_\theta\left(\nu', f\nu'\right)\left(1 + d_\theta\left(\mu', f\mu'\right)\right)}{1 + d_\theta\left(\mu', \nu'\right)}\right)^r \\ &\quad + \left(\frac{d_\theta\left(\nu', f\mu'\right)\left(1 + d_\theta\left(\mu', f\nu'\right)\right)}{1 + d_\theta\left(\mu', \nu'\right)}\right)^r\Bigg]^{1/r} \\ &= (\lambda_1 + \lambda_5)^{1/r} d_\theta\left(\mu', \nu'\right) < d_\theta\left(\mu', \nu'\right), \end{aligned} \tag{42}$$

which is a contradiction. Similarly, for case $r = 0$, we obtain $0 < d_\theta(\mu', \nu') < 0$, which also gives a contradiction. Thus, $\mu' = \nu'$; that is, f has a unique fixed point. □

Example 1. Let $\mathscr{X} = [0, 3]$ and $d : \mathscr{X} \times \mathscr{X} \longrightarrow [0,\infty)$ be such that $d(\mu, \nu) = |\mu - \nu|^2$ and $\theta(\mu, \nu) = \mu + \nu + 2$ for all $\mu, \nu \in \mathscr{X}$. Consider the mapping $f : \mathscr{X} \longrightarrow \mathscr{X}$ defined by

$$f(\mu) = \begin{cases} \dfrac{1}{3}, & \text{if } \mu \in [0, 1], \\ \dfrac{\mu}{3}, & \text{if } \mu \in (1, 3], \end{cases} \tag{43}$$

and

$$\alpha(\mu, \nu) = \begin{cases} 3, & \text{if } \mu, \nu \in [0, 1], \\ 1, & \text{if } \mu = 0, \nu = 3, \psi : [0,\infty) \longrightarrow [0,\infty) \text{defined by} \psi(t) = \dfrac{t}{3} \& \zeta(t, s) = \dfrac{s}{3} - t, \\ 0, & \text{otherwise.} \end{cases} \tag{44}$$

Note that

(i) $(\mathscr{X}, d_\theta)$ is an extended complete b-metric space with $\theta(\mu, \nu) = \mu + \nu + 2$

(ii) f is triangular α-orbital admissible mapping

(iii) For $\mu_0 \in [0, 1], f\mu_0 = 1/3 \in [0, 1]$, and therefore, $\alpha(\mu_0, f\mu_0) = 3 > 1$

(iv) f is continuous

(v) f^2 is continuous, where $f^2 = 1/3$

Furthermore, for $\mu = 1/3 \in \mathrm{Fix}_{f^2}(\mathscr{X})$, we get $\alpha(1/3, f1/3) = 3 > 1$

(vi) $\zeta(\alpha(\mu, \nu)d_\theta(f\mu, f\nu), \psi(\mathscr{P}^r_f(\mu, \nu))) \geq 0$

Consider $\mu, \nu \in [0, 1]$, then $f\mu = f\nu = 1/3$, and hence, $d_\theta(f\mu, f\nu) = 0$. For all $\mu, \nu \in [0, 1]$, we have

$$\begin{aligned} \zeta\left(\alpha(\mu, \nu)d_\theta(f\mu, f\nu), \psi\left(\mathscr{P}^r_f(\mu, \nu)\right)\right) &= \zeta\left(0, \psi\left(\mathscr{P}^r_f(\mu, \nu)\right)\right) \\ &= \frac{1}{3}\psi\left(\mathscr{P}^r_f(\mu, \nu)\right). \end{aligned} \tag{45}$$

Hence,

$$\zeta\left(\alpha(\mu, \nu)d_\theta(f\mu, f\nu), \psi\left(\mathscr{P}^r_f(\mu, \nu)\right)\right) \geq 1 \forall \mu, \nu \in [0, 1]. \tag{46}$$

Now, consider $\mu = 0, \nu = 3$, $r = 2$, and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1/5$, we have

$$\begin{aligned} &\zeta\left(\alpha(\mu, \nu)d_\theta(f\mu, f\nu), \psi\left(\mathscr{P}^r_f(\mu, \nu)\right)\right) = \zeta\left(\alpha(0, 3)d_\theta(f0, f3), \psi\left(\mathscr{P}^r_f(0, 3)\right)\right) \\ &= \frac{1}{3}\psi\left(\mathscr{P}^r_f(0, 3)\right) - \alpha(0, 3)d_\theta(f0, f3) \\ &= \frac{1}{3} \cdot \frac{1}{3}\left[\frac{1}{5}d^2_\theta(0, 3) + \frac{1}{5}d^2_\theta(0, f0) + \frac{1}{5}d^2_\theta(3, f3) + \frac{1}{5} \right. \\ &\quad \left. \cdot \left(\frac{d_\theta(3, f3)(1 + d_\theta(0, f0))}{1 + d_\theta(0, 3)}\right)^2 + \frac{1}{5}\left(\frac{d_\theta(3, f0)(1 + d_\theta(0, f3))}{1 + d_\theta(0, 3)}\right)^2\right]^{1/2} \\ &\quad - \alpha(0, 3)d_\theta\left(\frac{1}{3}, 1\right) = \frac{1}{9}\left[\frac{1}{5}\left(81 + \frac{1}{81} + 16 + \frac{16}{81} + \frac{64}{225}\right)\right]^{1/2} - \frac{4}{9} \geq 0. \end{aligned} \tag{47}$$

Hence,

$$\zeta\left(\alpha(0, 3)d_\theta(f0, f3), \psi\left(\mathscr{P}^r_f(0, 3)\right)\right) \geq 0. \tag{48}$$

In all other cases, we have $\alpha(\mu, \nu) = 0$, so

$$\zeta\left(0, \psi\left(\mathscr{P}^r_f(\mu, \nu)\right)\right) = \frac{1}{3}\psi\left(\mathscr{P}^r_f(\mu, \nu)\right) \geq 0. \tag{49}$$

Hence, we acquire that f is an admissible hybrid $\mathscr{Z}$-contraction. It follows all the hypothesis of Theorem 11, and so, $\mu = 1/3$ is the fixed point of f.

Let Φ be the collection of all auxiliary functions $\varphi : [0, \infty) \longrightarrow [0,\infty)$, which are continuous and $\varphi(\upsilon) = 0$ if and only if $\upsilon = 0$.

Corollary 12. *Let $(\mathscr{X}, d_\theta)$ be an extended b-metric space, $f : \mathscr{X} \longrightarrow \mathscr{X}$ and $\alpha : \mathscr{X} \times \mathscr{X} \longrightarrow [0,\infty)$. Suppose that there exist two functions $\varphi_1, \varphi_2 \in \Phi$ with $\varphi_1(\upsilon) < \upsilon < \varphi_2(\upsilon)$, for all $\upsilon > 0$, such that*

$$\varphi_2(\alpha(\mu, \nu)d_\theta(f\mu, f\nu)) \leq \varphi_1\left(\mathscr{P}^r_f(\mu, \nu)\right), \tag{50}$$

where ζ is defined as $\zeta(t, s) = \varphi_1(s) - \varphi_2(t)$. Additionally, suppose that

(i) f is triangular α-orbital admissible

(ii) There exists $\mu_0 \in \mathscr{X}$ such that $\alpha(\mu_0, f\mu_0) \geq 1$

(iii) Either f is continuous or

(iv) f^2 is continuous and $\alpha(\mu, f\mu) \geq 1$ for any $\mu \in \mathrm{Fix}_{f^2}(\mathscr{X})$

(v) If $\mu', \nu' \in \mathrm{Fix}_f(\mathscr{X})$, then $\alpha(\mu', \nu') \geq 1$

Then, f has a unique fixed point.

Corollary 13. *Let $(\mathscr{X}, d_\theta)$ be an extended b-metric space. Suppose that there exists a function $\varphi \in \Phi$, where $\Phi : [0,\infty) \longrightarrow [0,\infty)$ such that all $\varphi \in \Phi$ is continuous and $\varphi(\upsilon) = 0$ if and only if $\upsilon = 0$, such that*

$$\alpha(\mu,\nu)d_\theta(f\mu,f\nu)\le \mathscr{P}^r_f(\mu,\nu)-\varphi\left(\mathscr{P}^r_f(\mu,\nu)\right), \tag{51}$$

where ζ is defined as $\zeta(t,s)=s-\varphi(s)-t$. Furthermore, we assume that

(i) *f is triangular α-orbital admissible*

(ii) *There exists $\mu_0\in\mathscr{X}$ such that $\alpha(\mu_0,f\mu_0)\ge 1$*

(iii) *Either f is continuous or*

(iv) *f^2 is continuous and $\alpha(\mu,f\mu)\ge 1$ for any $\mu\in Fix_{f^2}(\mathscr{X})$*

(v) *If $\mu',\nu'\in Fix_f(\mathscr{X})$, then $\alpha(\mu',\nu')\ge 1$*

Then, f has a unique fixed point.

Corollary 14. *Let $(\mathscr{X},d_\theta)$ be an extended b-metric space. Suppose that there exist a function $\chi:[0,\infty)\longrightarrow[0,\infty)$ such that $\int_0^{\mathscr{P}^r_f(\mu,\nu)}\chi(p)dp$ exists and $\int_0^{\mathscr{P}^r_f(\mu,\nu)}\chi(p)dp>\varepsilon$, for every $\varepsilon>0$, with property that*

$$\alpha(\mu,\nu)d_\theta(f\mu,f\nu)\le\int_0^{\mathscr{P}^r_f(\mu,\nu)}\chi(p)dp, \tag{52}$$

where ζ is defined as $\zeta(t,s)=s-\int_0^t\chi(u)du$. Moreover, we suppose that

(i) *f is triangular α-orbital admissible*

(ii) *There exists $\mu_0\in\mathscr{X}$ such that $\alpha(\mu_0,f\mu_0)\ge 1$*

(iii) *Either f is continuous or*

(iv) *f^2 is continuous mapping and $\alpha(\mu,f\mu)\ge 1$ for any $\mu\in Fix_{f^2}(\mathscr{X})$*

(v) *If $\mu',\nu'\in Fix_f(\mathscr{X})$, then $\alpha(\mu',\nu')\ge 1$*

Then, f has a unique fixed point.

4. Application

In this section, we explore Ulam-Hyers stability and well posedness of the fixed point problem in the setup of an extended b-metric space (see [36] and references therein).

Definition 15. Let $f:\mathscr{X}\longrightarrow\mathscr{X}$ be a self-mapping defined on an extended b-metric space. Consider the fixed point problem

$$\mu=f\mu. \tag{53}$$

The fixed point problem (53) is well-posed if

(i) $\mathrm{Fix}_f(\mathscr{X})=\{\mu'\}$

(ii) If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence such that $d_\theta(\mu_n,f\mu_n)\longrightarrow 0$, as $n\longrightarrow\infty$, then $\mu_n\longrightarrow\mu'$, as $n\longrightarrow\infty$

Theorem 16. *Let $(\mathscr{X},d_\theta)$ be an extended b-metric space. Suppose that all the assumptions of Theorem 11 hold, and $r>0$. Additionally, we assume that for any sequence $(\mu_n)_{n\in\mathbb{N}}$, $d_\theta(\mu_n,f\mu_n)\longrightarrow 0$, as $n\longrightarrow\infty$, we have $\alpha(\mu_n,\mu')\ge 1$, for all $n\in\mathbb{N}$, where $\mu'\in Fix_f(\mathscr{X})$. If $\lambda_1+\lambda_5<1/\eta^2(r)$, where $\eta(r)=\max\{1,2^{r-1}(\theta(\mu,\nu))^r\}\forall\mu,\nu\in\mathscr{X}$, then the fixed point problem (53) is well-posed.*

Proof. As $\mu'=\mathrm{Fix}_f(\mathscr{X})$, by equation (7),

$$\begin{aligned}0&\le\zeta\left(\alpha\left(\mu_n,\mu'\right)d_\theta\left(f\mu_n,f\mu'\right),\psi\left(\mathscr{P}^r_f\left(\mu_n,\mu'\right)\right)\right)\\&<\psi\left(\mathscr{P}^r_f\left(\mu_n,\mu'\right)\right)-\alpha\left(\mu_n,\mu'\right)d_\theta\left(f\mu_n,f\mu'\right).\end{aligned} \tag{54}$$

Consider

$$\begin{aligned}d_\theta\left(\mu_n,\mu'\right)&\le\theta\left(\mu_n,\mu'\right)\left[d_\theta(\mu_n,f\mu_n)+d_\theta\left(f\mu_n,\mu'\right)\right]\\
&=\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)d_\theta\left(f\mu_n,\mu'\right)\\
&\le\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)\alpha\left(\mu_n,\mu'\right)d_\theta\\
&\quad\cdot\left(f\mu_n,f\mu'\right)\le\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)\\
&\quad+\theta\left(\mu_n,\mu'\right)\psi\left(\mathscr{P}^r_f\left(\mu_n,\mu'\right)\right)\\
&<\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)\mathscr{P}^r_f\left(\mu_n,\mu'\right)\\
&\le\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)\\
&\quad\cdot\left[\lambda_1d^r_\theta\left(\mu_n,\mu'\right)+\lambda_2d^r_\theta(\mu_n,f\mu_n)+\lambda_3d^r_\theta\left(\mu',f\mu'\right)\right.\\
&\quad+\lambda_4\left(\frac{d_\theta\left(\mu',f\mu'\right)(1+d_\theta(\mu_n,f\mu_n))}{1+d_\theta\left(\mu_n,\mu'\right)}\right)^r\\
&\quad\left.+\lambda_5\left(\frac{d_\theta\left(\mu',f\mu_n\right)\left(1+d_\theta\left(\mu_n,f\mu'\right)\right)}{1+d_\theta\left(\mu_n,\mu'\right)}\right)^r\right]^{1/r}\\
&=\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)\\
&\quad\cdot\left[\lambda_1d^r_\theta\left(\mu_n,\mu'\right)+\lambda_2d^r_\theta(\mu_n,f\mu_n)+\lambda_5d^r_\theta\left(\mu',f\mu_n\right)\right]^{1/r}\\
&\le\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)\\
&\quad\cdot\left[\lambda_1d^r_\theta\left(\mu_n,\mu'\right)+\lambda_2d^r_\theta(\mu_n,f\mu_n)+\left(\theta\left(\mu',f\mu_n\right)\right)^r\lambda_5\right.\\
&\quad\left.\cdot\left(d_\theta\left(\mu',\mu_n\right)+d_\theta(\mu_n,f\mu_n)\right)^r\right]^{1/r}\\
&\le\theta\left(\mu_n,\mu'\right)d_\theta(\mu_n,f\mu_n)+\theta\left(\mu_n,\mu'\right)\\
&\quad\cdot\left[\lambda_1d^r_\theta\left(\mu_n,\mu'\right)+\lambda_2d^r_\theta(\mu_n,f\mu_n)+2^{r-1}\right.\\
&\quad\cdot\left(\theta\left(\mu',f\mu_n\right)\right)^r\lambda_5d^r_\theta\left(\mu',\mu_n\right)+2^{r-1}\\
&\quad\left.\cdot\left(\theta\left(\mu',f\mu_n\right)\right)^r\lambda_5d^r_\theta(\mu_n,f\mu_n)\right]^{1/r}.\end{aligned} \tag{55}$$

In this way, we obtain

$$\begin{aligned} d_\theta^r\left(\mu_n,\mu'\right) \le{}& 2^{r-1}\left(\theta\left(\mu_n,\mu'\right)\right)^r d_\theta^r(\mu_n,f\mu_n) + 2^{r-1} \\ &\cdot \left(\theta\left(\mu_n,\mu'\right)\right)^r \lambda_1 d_\theta^r\left(\mu_n,\mu'\right) + 2^{r-1} \\ &\cdot \left(\theta\left(\mu_n,\mu'\right)\right)^r \lambda_2 d_\theta^r(\mu_n,f\mu_n) + 2^{2r-2} \\ &\cdot \left(\theta\left(\mu_n,\mu'\right)\right)^r \left(\theta\left(\mu',f\mu_n\right)\right)^r \lambda_5 d_\theta^r\left(\mu',\mu_n\right) \\ &+ 2^{2r-2}\left(\theta\left(\mu_n,\mu'\right)\right)^r (\theta(\mu_n,f\mu_n))^r \lambda_5 d_\theta^r(\mu_n,f\mu_n), \end{aligned} \tag{56}$$

or we can write

$$\begin{aligned} &\left[1-2^{r-1}\left(\theta\left(\mu_n,\mu'\right)\right)^r\lambda_1 - 2^{2r-2}\left(\theta\left(\mu_n,\mu'\right)\right)^r\left(\theta\left(\mu',f\mu_n\right)\right)^r\lambda_5\right] d_\theta^r\left(\mu_n,\mu'\right) \\ &\quad\le 2^{r-1}\left(\theta\left(\mu_n,\mu'\right)\right)^r\left[1+\lambda_2+2^{r-1}(\theta(\mu_n,f\mu_n))^r\lambda_5\right] d_\theta^r(\mu_n,f\mu_n). \end{aligned} \tag{57}$$

From here, we get

$$\begin{aligned} &d_\theta^r\left(\mu_n,\mu'\right) \\ &\quad\le \frac{2^{r-1}\left(\theta\left(\mu_n,\mu'\right)\right)^r\left[1+\lambda_2+2^{r-1}(\theta(\mu_n,f\mu_n))^r\lambda_5\right]}{\left[1-2^{r-1}\left(\theta\left(\mu_n,\mu'\right)\right)^r\lambda_1-2^{2r-2}\left(\theta\left(\mu_n,\mu'\right)\right)^r\left(\theta\left(\mu',f\mu_n\right)\right)^r\lambda_5\right]} d_\theta^r(\mu_n,f\mu_n), \end{aligned} \tag{58}$$

$$d_\theta^r\left(\mu_n,\mu'\right) \le \frac{\eta(r)[1+\lambda_2+\eta(r)\lambda_5]}{[1-\eta(r)\lambda_1-\eta^2(r)\lambda_5]} d_\theta^r(\mu_n,f\mu_n). \tag{59}$$

As $n\longrightarrow\infty$, we have that $\lim_{n\longrightarrow\infty} d_\theta^r(\mu_n,f\mu_n)=0$. So, $\lim_{n\longrightarrow\infty} d_\theta^r(\mu',\mu_n)=0$.

Thus, the fixed point problem (53) is well-posed. □

Definition 17. The fixed point problem $\mu=f\mu$ is called generalized Ulam-Hyers stable if and only if there exists $\omega:[0,\infty)\longrightarrow[0,\infty)$ which is increasing, continuous at 0 with $\omega(0)=0$, such that for every $\xi>0$ and for each $\nu'\in\mathscr{X}$,

$$d_\theta(\nu,f\nu)\le\xi, \tag{60}$$

there exists a solution μ' of the fixed point problem such that $d_\theta(\nu',\mu')\le\omega(\xi)$.

If there exists $x>0$ such that $\omega(a)\coloneqq x.a$, for each $a\in\mathbb{R}^+$, then the fixed point problem is referred to be Ulam-Hyers stable.

Theorem 18. *Let $(\mathscr{X},d_\theta)$ be an extended b-metric space. Suppose that all the assumptions of Theorem 11 hold and $r>0$.*

Furthermore, we assume that $\alpha(\nu',\mu')\ge 1$, for all $\nu'\in\mathscr{X}$ satisfying (60), where $\mu'\in Fix_f(\mathscr{X})$. If $\lambda_1+\lambda_5<1/\eta^2(r)$, where $\eta(r)=\max\{1,2^{r-1}(\theta(\mu,\nu))^r\}\forall\mu,\nu\in\mathscr{X}$, then the fixed point problem $\mu=f\mu$ is Ulam-Hyers stable.

Proof. By (7),

$$\begin{aligned} 0\le\zeta&\left(\alpha\left(\nu',\mu'\right)d_\theta\left(f\nu',f\mu'\right),\psi\left(\mathscr{P}_f^r\left(\nu',\mu'\right)\right)\right) \\ &<\psi\left(\mathscr{P}_f^r\left(\nu',\mu'\right)\right)-\alpha\left(\nu',\mu'\right)d_\theta\left(f\nu',f\mu'\right). \end{aligned} \tag{61}$$

Consider

$$\begin{aligned} d_\theta\left(\nu',\mu'\right) &= d_\theta\left(\nu',f\mu'\right)\le\theta\left(\nu',\mu'\right) \\ &\quad\cdot\left[d_\theta\left(\nu',f\mu'\right)+d_\theta\left(f\nu',f\mu'\right)\right] \\ &=\theta\left(\nu',\mu'\right)d_\theta\left(\nu',f\mu'\right)+\theta\left(\nu',\mu'\right)d_\theta\left(f\nu',f\mu'\right) \\ &\le\theta\left(\nu',\mu'\right)d_\theta\left(\nu',f\nu'\right)+\theta\left(\nu',\mu'\right)\alpha\left(\nu',\mu'\right)d_\theta \\ &\quad\cdot\left(f\nu',f\mu'\right)\le\theta\left(\nu',\mu'\right)\xi+\theta\left(\nu',\mu'\right)\psi\left(\mathscr{P}_f^r\left(\nu',\mu'\right)\right) \\ &<\theta\left(\nu',\mu'\right)\xi+\theta\left(\nu',\mu'\right)\mathscr{P}_f^r\left(\nu',\mu'\right) \\ &\le\theta\left(\nu',\mu'\right)\xi+\theta\left(\nu',\mu'\right) \\ &\quad\cdot\left[\lambda_1 d_\theta^r\left(\nu',\mu'\right)+\lambda_2 d_\theta^r\left(\nu',f\nu'\right)+\lambda_3 d_\theta^r\left(\mu',f\mu'\right)\right. \\ &\quad+\lambda_4\left(\frac{d_\theta\left(\mu',f\mu'\right)\left(1+d_\theta\left(\nu',f\nu'\right)\right)}{1+d_\theta\left(\nu',\mu'\right)}\right)^r \\ &\quad\left.+\lambda_5\left(\frac{d_\theta\left(\mu',f\nu'\right)\left(1+d_\theta\left(\nu',f\mu'\right)\right)}{1+d_\theta\left(\nu',\mu'\right)}\right)^r\right]^{1/r} \\ &=\theta\left(\nu',\mu'\right)\xi+\theta\left(\nu',\mu'\right) \\ &\quad\cdot\left[\lambda_1 d_\theta^r\left(\nu',\mu'\right)+\lambda_2\xi^r+\lambda_5 d_\theta^r\left(\mu',f\nu'\right)\right]^{1/r} \\ &\le\theta\left(\nu',\mu'\right)\xi+\theta\left(\nu',\mu'\right)\left[\lambda_1 d_\theta^r\left(\nu',\mu'\right)+\lambda_2\xi^r\right. \\ &\quad\left.+\left(\theta\left(\mu',f\nu'\right)\right)^r\lambda_5\left(d_\theta\left(\mu',\nu'\right)+d_\theta\left(\nu',f\mu'\right)\right)^r\right]^{1/r} \\ &\le\theta\left(\nu',\mu'\right)\xi+\theta\left(\nu',\mu'\right)\left[\lambda_1 d_\theta^r\left(\nu',\mu'\right)+\lambda_2\xi^r+2^{r-1}\right. \\ &\quad\cdot\left(\theta\left(\mu',f\nu'\right)\right)^r\lambda_5 d_\theta^r\left(\mu',\nu'\right)+2^{r-1}\left(\theta\left(\mu',f\nu'\right)\right)^r\lambda_5 d_\theta^r \\ &\quad\left.\cdot\left(\nu',f\nu'\right)\right]^{1/r}. \end{aligned} \tag{62}$$

Thus, we get

$$\begin{aligned} d_\theta^r\left(\nu',\mu'\right)\le{}&2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\xi^r+2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\lambda_1 d_\theta^r \\ &\cdot\left(\nu',\mu'\right)+2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\lambda_2\xi^r+2^{2r-2} \\ &\cdot\left(\theta\left(\nu',\mu'\right)\right)^r\left(\theta\left(\mu',f\nu'\right)\right)^r\lambda_5 d_\theta^r\left(\mu',\nu'\right) \\ &+2^{2r-2}\left(\theta\left(\nu',\mu'\right)\right)^r\left(\theta\left(\nu',f\nu'\right)\right)^r\lambda_5\xi^r, \end{aligned} \tag{63}$$

or we can write

$$\left[1-2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\lambda_1-2^{2r-2}\left(\theta\left(\nu',\mu'\right)\right)^r\left(\theta\left(\mu',f\nu'\right)\right)^r\lambda_5\right]d_\theta^r \cdot \left(\mu',\nu'\right) \le 2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\left[1+\lambda_2+2^{r-1}\left(\theta\left(\nu',f\nu'\right)\right)^r\lambda_5\right]\xi^r. \tag{64}$$

Therefore, we obtain

$$d_\theta^r\left(\mu',\nu'\right) \le \frac{2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\left[1+\lambda_2+2^{r-1}\left(\theta\left(\nu',f\nu'\right)\right)^r\lambda_5\right]}{\left[1-2^{r-1}\left(\theta\left(\nu',\mu'\right)\right)^r\lambda_1-2^{2r-2}\left(\theta\left(\nu',\mu'\right)\right)^r\left(\theta\left(\mu',f\nu'\right)\right)^r\lambda_5\right]}\xi^r. \tag{65}$$

Thus, we get

$$d_\theta^r\left(\mu',\nu'\right) \le \frac{\eta(r)[1+\lambda_2+\eta(r)\lambda_5]}{[1-\eta(r)\lambda_1-\eta^2(r)\lambda_5]}\xi^r. \tag{66}$$

Hence,

$$d_\theta\left(\mu',\nu'\right) \le x\xi^r, \tag{67}$$

where $x=(\eta(r)[1+\lambda_2+\eta(r)\lambda_5])/([1-\eta(r)\lambda_1-\eta^2(r)\lambda_5])$ for all $r>0$ and $\lambda_1,\lambda_5\in[0,1)$. □

5. Conclusion

In this research paper, we consolidated and refined several existing results in literature by bringing up the notation of admissible hybrid $\mathcal{Z}$-contractions in the setup of an extended b-metric space. Accordingly, all the exhibited results are authentic in context of complete b-metric spaces by letting $\theta(\mu,\nu)=\delta$, where $\delta\ge 1$, and in context of complete metric spaces by letting $\delta=1$. Furthermore, the paper generalizes the results of [36, 45, 46, 51]. Numerous fixed point results can be concluded in standard b-metric spaces via a partial order or a cyclic contraction. Moreover, one can derive results in extended b-metric spaces using [52–55].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

The statements made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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