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On Perfect Commutative $EIFA$-rings.

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Abstract

We consider the class $\mathcal{F}$ of endo-Artinian modules, i.e. the modules M which satisfying the descending chain condition for endomorphic images: any descending chain $Imf_1 \supset Imf_2 \supset \cdots \supset Imf_n \supset \cdots$ is stationary, where $f_i \in End(M)$. Let $\mathcal{A}$ be the class of Artinian modules. It is clear that every Artinian R-module M is endo-Artinian, so $\mathcal{A} \subset \mathcal{F}$, but the converse is not true. Indeed, $\mathbb{Q}$ is a non-Artinian $\mathbb{Z}$-module which is endo-Artinian. The aim of this work, is to characterize perfect commutative rings for which $\mathcal{F}$ and $\mathcal{A}$ are identical.

Keywords: Artinian; $EIFA$-ring; endo-Artinian; perfect.

1 Introduction

The pioneering work on perfect rings was done by Hyman Bass [1]. A ring R is left perfect (resp. right perfect) in case each of its left (resp. right) modules has a projective cover. The descending chain condition on ideals was introduced by Artin. Rings in which any descending chain of ideals is finite are called Artinian. A. Kaidi introduced in 2009 in [2] the class of endo-Noetherian (resp. endo-Artinian) modules. Endo-Artinian concept is the dual property of endo-Noetherian notion. Recently in 2013, M.A. Ndiaye and C.T. Gueye characterized partially $EKFN$-rings in [3]. The purpose of this paper is to characterize rings on which an endo-Artinian module M is Artinian. Such rings will be called $EIFA$-rings. We prove that a perfect commutative ring is an $EIFA$-ring if and only if it is an Artinian principal ideal ring.
Our paper is structured as follows: the first section covers the properties and basic aspects of certain classes of modules and rings. In the second section, we characterize perfect commutative $EIFA$-rings.

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2 Preliminaries

The rings considered in this section are commutative with unit. Unless otherwise mentioned, all the modules considered are left unitary modules.

Definition 2.1. A ring R is called perfect if R is both right and left perfect.

Lemma 2.1. *([4], P.249)*
Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C-endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$, then:

1. $r\sigma = \sigma^2 = 0$;
2. *If F is a C-endomorphism of M commuting with σ, then for all $m \in M$, $F(rm) = rmF(1)$.*

Proposition 2.1. *([4], P.250)*
Let R be an Artinian ring having at least one non-principal ideal. Then there exists an R-module M which is not finitely generated.

Proposition 2.2. *([5], P.255-256)*
Let M be a module over a left Artinian ring R. Then the following properties of M are equivalent:

1. *M is finitely generated;*
2. *M is Artinian.*

Definition 2.2. An R-module M is called endo-Artinian if any descending chain of endomorphic images $Imf_1 \supset Imf_2 \supset \cdots \supset Imf_n \supset \cdots$ stabilizes, where f_i's are endomorphisms of M, i.e. there exists a positive integer n such that $Imf_n = Imf_{n+1}$.

Proposition 2.3. *([2], P.5)*
Let R be a ring. Then the following statements are equivalent:

1. *R is endo-Artinian;*
2. *R satisfies the descending chain condition for principal left ideals;*
3. *R is right perfect.*

Proposition 2.4. *([3], P.429)*
Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C-endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$. Then every non-zero C-endomorphism F of M commuting with σ is such that either $KerF = rM$, or is a monomorphism.

Theorem 2.2. ***Fundamental theorem of isomorphism*** *([6] P.54)*
Let $f : M \longrightarrow N$ be an R-homomorphism. Then, f gives rise to an isomorphism of R-modules $\tilde{f} : M/Kerf \xrightarrow{\sim} Imf$.

3 The Main Results

Let R be a commutative ring with identity $1 \neq 0$.

Definition 3.1. R is said to be an $EIFA$-ring[1] if every endo-Artinian R-module is Artinian.

[1] $EIFA$ is derived from the two concepts namely Endo Image Finite and Artinian.

Example 3.1. *A semisimple ring and an Artinian ring with principal ideal, are $EIFA$-rings.*

Theorem 3.2. *Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C-endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$. Then every non-zero C-endomorphism F of M commuting with σ is such that either $M/rM \simeq ImF$, or is an epimorphism.*

Proof. We know by Proposition 2.4 that every non-zero C-endomorphism F of M commuting with σ is such that either $KerF = rM$, or is a monomorphism.
Assume that $KerF = rM$, then $M/rM \simeq ImF$ by Theorem 2.2.
Conversely, if F is a monomorphism, so $F(1)$ is invertible, then for any element $m \in M$, we have $F(rmF(1)^{-1}) = rmF(1)^{-1}F(1) = rm$ by Lemma 2.1. Hence $F(rM) = rM$ $(*)$. Let $m \in M$, then there exists an element $m' \in M$, such that $F(rm') = rm$ by $(*)$. Thus $r(F(m') - m) = 0$. It follows that $(F(m') - m) \in rM$. As $rM \subset ImF$, by $(*)$, we deduce that $m \in ImF$. It follows that F is an epimorphism. □

Theorem 3.3. *Let R be an Artinian ring having at least one non-principal ideal. Then there exists an endo-Artinian R-module M which is non-Artinian.*

Proof. By Proposition 2.1, there exists an R-module M which is not finitely generated. Then M is non-Artinian. Let $Imf_1 \supset Imf_2 \supset \cdots \supset Imf_n \supset \cdots$ be a descending chain for R-endomorphic images, where $f_i \in End_R(M)$. We know that R-endomorphisms of M are C-endomorphisms of M which commute with σ, then any descending chain for R-endomorphic images stabilizes. Thus M is endo-Artinian. □

Theorem 3.4. ***Characterization theorem***
Let R be a perfect ring, the following conditions are equivalent:

1. *R is an $EIFA$-ring;*
2. *R is an Artinian principal ideal ring.*

Proof.

1⇒ 2) Assume that R is an $EIFA$-ring. Since R is perfect, then by Proposition 2.3, R is endo-Artinian, so R is Artinian. Assume that R has at least one non-principal ideal, so there exists an endo-Artinian module which is non-Artinian by Theorem 3.3, contradiction. Thus R is an Artinian principal ideal ring;

2⇒ 1) Assume that R is an Artinian principal ideal ring.Then, R is an $EIFA$-ring.

□

4 Conclusion

Let R be a commutative ring with identity $1 \neq 0$. Every Artinian R-module M is endo-Artinian, but the converse is not true. R is said to be an $EIFA$-ring if every endo-Artinian R-module is Artinian. We have studied and characterized partially in this paper $EIFA$-ring. Authors think that a characterization of $EIFA$-ring in nonsingular ring can be very interesting. Finally, studying the relationship between $EIFA$-rings and $EKFN$-rings by showing which one implies the other or what the conditions to be imposed in order to satisfy this property can be exceptional.

Competing Interests

The authors declare that no competing interests exist.

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