British Journal of Mathematics & Computer Science
4(8): 1156-1165, 2014

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New Bounds for Restricted Isometry Constant for the s-sparse Recovery via Compressed Sensing

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Original Research Article

Received: 11 January 2014
Accepted: 09 February 2014
Published: 22 February 2014

Abstract

The main purpose of this paper is to establish the sufficient condition for the restricted isometry constant δ_s in compressed sensing by using T. Cai and A. Zhang idea. Let $\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}$ and $\boldsymbol{h} = (h_1, h_2, \cdots, h_n)$, where $\boldsymbol{x}$ is an unknown signal and $\boldsymbol{x}^\star$ is the CS-solution. For simplicity, we assume that the index of $\boldsymbol{h}$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_n|$. Let s be a fixed positive integer, $T_0 = \{1, 2, \cdots, s\}$ and $T_1 \subset T_0$. In this paper, we focus the quality of $\boldsymbol{h}_{T_0}$ and research good conditions for the recovery of sparse signals by investigating the difference between $\|\boldsymbol{h}_{T_1}\|_1$ and $\|\boldsymbol{h}_{T_1^c}\|_1$. We shall show that if $\delta_s < 0.5$ under an assumption for $\|\boldsymbol{h}_{T_1}\|_1$, and similarly if $\delta_{\frac{3}{4}s} < 0.414$ or $\delta_{\frac{24}{25}s} < 0.436$, then we have stable recovery of approximately sparse signals.

Keywords: Compressed sensing, Restricted isometry constants, Restricted isometry property, Sparse approximation, Sparse signal recovery.

1 Introduction

This paper introduces the theory of compressed sensing(CS). For a signal $\boldsymbol{x} \in \boldsymbol{R}^n$, let $\|\boldsymbol{x}\|_1$ be l_1 norm of $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_2$ be l_2 norm of $\boldsymbol{x}$. Let $\boldsymbol{x}$ be a sparse or nearly sparse vector. Compressed sensing aims to recover high-dimensional signal (for example: images signal, voice signal, code signal...etc.) from only a few samples or linear measurements. Efficient recovery of sparse signals has been a very active field in applied mathematics, statistics, machine learning and signal processing. Formally, one considers the following model:

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{1.1}$$

where A is a $m \times n$ matrix($m < n$) and $\boldsymbol{z}$ is an unknown noise term.

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Our goal is to reconstruct an unknown signal $\boldsymbol{x}$ based on A and $\boldsymbol{y}$ are given. Then we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1, \quad \text{subject to } \|\boldsymbol{y} - A\boldsymbol{x}\|_2 \leq \varepsilon, \tag{1.2}$$

where ε is an upper bound on the the size of the noisy contribution.
In fact, a crucial issue is to research good conditions under which the inequality

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq C_0 \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + C_1 \varepsilon, \tag{1.3}$$

for some suitable constants C_0 and C_1, where T_0 is any location of $\{1, 2, \cdots, n\}$ with number $|T_0| = s$ of elements of T_0 and $\boldsymbol{x}_{T_0}$ is the restriction of $\boldsymbol{x}$ to indices in T_0. One of the most generally known condition for CS theory is the restricted isometry property(RIP) introduced by [1]. When we discuss our proposed results, it is an important notion. The RIP needs that the subsets of columns of A for all locations in $\{1, 2, \cdots, n\}$ behave nearly orthonormal system. In detail, a matrix A satisfies the RIP of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1 - \delta)\|\boldsymbol{a}\|_2^2 \leq \|A\boldsymbol{a}\|_2^2 \leq (1 + \delta)\|\boldsymbol{a}\|_2^2 \tag{1.4}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be an s-sparse vector if it has at most s nonzero entries. The minimum δ satisfying the above restrictions is said to be the restricted isometry constant and is denoted by δ_s.
Many researchers has been shown that l_1 optimization can recover an unknown signal in noiseless case and noisy case under various sufficient conditions on δ_s or δ_{2s} when A obeys the RIP. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered [1]. Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ [2]. Others, $\delta_{2s} < 0.4652$ is used by [3], $\delta_{2s} < 0.4721$ for cases such that s is a multiple of 4 or s is very large by [4], $\delta_{2s} < 0.4734$ for the case such that s is very large by [3] and $\delta_s < 0.307$ by [4]. In a resent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ for the special case such that $n \leq 4s$ [5]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_s < 0.333$ for general case [6]. T. Cai and A. Zhang have improved the sufficient condition to δ_k in case of $k \geq \frac{4}{3}s$ in particular $\delta_{2s} < 0.707$ [7]. H. Inoue has obtained the sufficient conditions of $\tilde{\delta}_s < 0.5$ and $\tilde{\delta}_{2s} < 0.828$ by using rescaling method [8].
Let $\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}$ and $\boldsymbol{h} = (h_1, h_2, \cdots, h_n)$, where $\boldsymbol{x}$ is an unknown signal and $\boldsymbol{x}^\star$ is the CS-solution. For simplicity, we assume that the index of $\boldsymbol{h}$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_n|$. Let s be a fixed positive integer, $T_0 = \{1, 2, \cdots, s\}$ and $T_1 \subset T_0$. As stated above, the present best sufficient condition for the restricted isometry constant of order s is $\delta_s < 0.333$. In this paper, we shall improve the sufficient condition for δ_s by investigating the difference of the l_1-norm $\|\boldsymbol{h}_{T_1}\|_1$ and the l_1-norm $\|\boldsymbol{h}_{T_1^c}\|_1$, where $T_1 = \{1, 2, \cdots, \frac{s}{2}\}$. In more details, in Theorem 2.1 it is shown under the assumption that A obeys the RIP of order s and $\delta_s < \frac{1}{2}$ that if

$$\min\{|h_i|;\ i \in T_1\} \geq \frac{\|\boldsymbol{h}_{T_1^c}\|_1}{s/2}, \tag{1.5}$$

then

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq C_1 \varepsilon, \tag{1.6}$$

if otherwise, then

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq C_0 \|\boldsymbol{x} - \boldsymbol{x}_{\frac{s}{2}}\|_1 + C_1 \varepsilon. \tag{1.7}$$

This means that in case of noiseless, if (1.5) holds, then $\boldsymbol{x}$ is completely recovered as $\boldsymbol{x}^\star$ and if (1.5) does not hold, then every $\frac{s}{2}$-sparse vector $\boldsymbol{x}$ is completely recovered as $\boldsymbol{x}^\star$. This result shows that the sufficient condition for δ_s can be substantially improved, but the condition for the sparsity becomes worse. On the other hand, any sufficient conditions for $\delta_{s'}$ $(s' < s)$ have never been given. By changing a subset T_1 of T_0 or the condition (1.5), we shall similarly give sufficient conditions for $\delta_{s'}$ $(s' < s)$ in Theorem 2.2 and Theorem 2.3.

Our analysis is very simple and elementary. We introduce the proposed results using the T. Cai and A. Zhang idea and H. Inoue idea. We regard Theorem 2.1, Theorem 2.2 and Theorem 2.3 as the main results in this paper. Otherwise, in Section 2, we prepare some notions and lemmas to prove the main theorems, and we introduce new bounds of δ_s and $\delta_{s'}$ $(s' < s)$.

2 Main Theorem

2.1 Preliminaries and Some Lemmas

We first prepare three lemmas needed for the proof of Theorem 2.1, Theorem 2.2 and Theorem 2.3.

The following result plays an important role in this paper.
Lemma 2.1. For a positive number α and a positive integer k, define the polytope $T(\alpha, k) \subset \boldsymbol{R}^p$ by

$$T(\alpha, k) = \{\boldsymbol{v} \in \boldsymbol{R}^p;\ \|\boldsymbol{v}\|_\infty \leq \alpha,\ \|\boldsymbol{v}\|_1 \leq k\alpha\}. \tag{2.1}$$

For any $\boldsymbol{v} \in \boldsymbol{R}^p$, define the set of sparse vectors $U(\alpha, k, \boldsymbol{v}) \subset \boldsymbol{R}^p$ by

$$\begin{aligned} U(\alpha, k, \boldsymbol{v}) = \{&\boldsymbol{u} \in \boldsymbol{R}^p;\ \mathrm{supp}(\boldsymbol{u}) \subseteq \mathrm{supp}(\boldsymbol{v}), \\ &\|\boldsymbol{u}\|_0 \leq k,\ \|\boldsymbol{u}\|_1 = \|\boldsymbol{v}\|_1,\ \|\boldsymbol{u}\|_\infty \leq \alpha\}. \end{aligned} \tag{2.2}$$

Then $\boldsymbol{v} \in T(\alpha, k)$ if and only if $\boldsymbol{v}$ is in the convex hull of $U(\alpha, k, \boldsymbol{v})$. In particular, any $\boldsymbol{v} \in T(\alpha, k)$ can be expressed as

$$\begin{aligned} \boldsymbol{v} &= \sum_{i=1}^{N} \lambda_i \boldsymbol{u}_i, \quad 0 \leq \lambda_i \leq 1, \\ \sum_{i=1}^{N} \lambda_i &= 1, \quad \boldsymbol{u}_i \in U(\alpha, k, \boldsymbol{v}). \end{aligned} \tag{2.3}$$

Proof. The proof of this lemma can be obtained by [[7], Lemma 1.1].

Suppose that A obeys the RIP of order s. Then the following is easily shown.

Lemma 2.2. Let s' and s'' be positive integers with $s' + s'' \leq s$. Then

$$\left|\langle A\boldsymbol{a}',\ A\boldsymbol{a}''\rangle\right| \leq \delta_s \|\boldsymbol{a}'\|_2 \|\boldsymbol{a}''\|_2$$

for any s'-sparse vector $\boldsymbol{a}'$ and s''-sparse vector $\boldsymbol{a}''$ in $\boldsymbol{R}^n$ with disjoint supports.

Suppose that $\boldsymbol{x}$ is an original signal we need to recover and $\boldsymbol{x}^\star$ is the solution of CS optimization problem (1.2). Let $\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}$ and $\boldsymbol{h} = (h_1, h_2, \cdots, h_n)$. For simplicity, we may assume that the index of $\boldsymbol{h}$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_n|$. By (1.2) we have

$$\|A\boldsymbol{h}\|_2 \leq 2\varepsilon. \tag{2.4}$$

By the definitin of CS optimization (1.2), we have the following result. For the proof refer to [2].

Lemma 2.3. Let s' be positive integer and $T' = \{1, 2, \cdots, s'\}$. Then

$$\|\boldsymbol{h}_{T'^c}\|_1 \leq \|\boldsymbol{h}_{T'}\|_1 + 2\|\boldsymbol{x} - \boldsymbol{x}_{s'}\|_1, \tag{2.5}$$

where $\boldsymbol{x}_{s'}$ is the vector consisting of the s' largest entries of $\boldsymbol{x}$ in magnitude.

Throughout this paper let s be a fixed positive integer and let $T_0 = \{1, 2, \cdots, s\}$.

2.2 Bound for δ_s

Let $T_1 = \{1, 2, \cdots, \frac{s}{2}\}$. Then we have the following:

Theorem 2.1. Assume that A obeys the RIP of order s and $\delta_s < \frac{1}{2}$. Then if

$$\min\{|h_i|;\ \ i \in T_1\} \geq \frac{\|\boldsymbol{h}_{T_1^c}\|_1}{s/2}, \tag{2.6}$$

then

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq \frac{4\sqrt{1+\delta_{\frac{s}{2}}}}{1-2\delta_s}\varepsilon. \tag{2.7}$$

If otherwise, then

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq \frac{2\sqrt{2}(3-2\delta_s)}{\sqrt{s}(1-2\delta_s)}\|\boldsymbol{x} - \boldsymbol{x}_{\frac{s}{2}}\|_1 + \frac{4\sqrt{1+\delta_{\frac{s}{2}}}}{1-2\delta_s}\varepsilon. \tag{2.8}$$

Proof. Suppose that (2.6) holds. We put $\alpha = \frac{\|\boldsymbol{h}_{T_1^c}\|_1}{s/2}$ and consider a decomposition $\{T_2, T_3\}$ of T_1^c as follows:

$$\begin{aligned} T_2 &= \{i \in T_1^c;\ \ |h_i^{T_1^c}| > \alpha\}, \\ T_3 &= \{i \in T_1^c;\ \ |h_i^{T_1^c}| \leq \alpha\}, \end{aligned}$$

where h_i^T denotes the i-component of $\boldsymbol{h}$ for a location T of $\{1, 2, \cdots, n\}$. Then we have

$$\alpha|T_2| < \|\boldsymbol{h}_{T_2}\|_1 \leq \|\boldsymbol{h}_{T_1^c}\|_1 = \frac{1}{2}s\alpha, \tag{2.9}$$

and so

$$r \equiv |T_2| \leq \frac{1}{2}s. \tag{2.10}$$

Furthermore, we have

$$\|\boldsymbol{h}_{T_3}\|_\infty \leq \alpha$$

and by (2.9)

$$\begin{aligned} \|\boldsymbol{h}_{T_3}\|_1 &= \|\boldsymbol{h}_{T_1^c}\|_1 - \|\boldsymbol{h}_{T_2}\|_1 \\ &\leq \frac{1}{2}s\alpha - \alpha r \\ &= \alpha\left(\frac{1}{2}s - r\right). \end{aligned}$$

Using Lemma 2.1 for $k = \frac{1}{2}s - r$, there exist $\{\lambda_i\}_{1\leq i\leq N}$ and $\{\boldsymbol{u}_i\}_{1\leq i\leq N}$ such that

$$\boldsymbol{h}_{T_3} = \sum_{i=1}^{N} \lambda_i \boldsymbol{u}_i, \tag{2.11}$$

where

$$\begin{aligned} 0 \le \lambda_i \le 1 \quad &, \quad \sum_{i=1}^{N} \lambda_i = 1 \\ \text{supp } \boldsymbol{u}_i \quad &\subset \quad T_3, \qquad (2.12) \\ |\text{supp } \boldsymbol{u}_i| \quad &\le \quad \frac{1}{2}s - r, \qquad (2.13) \\ \|\boldsymbol{u}_i\|_\infty \quad &\le \quad \alpha, \end{aligned}$$

and so

$$\begin{aligned} \|\boldsymbol{u}_i\|_2 \quad &\le \quad \|\boldsymbol{u}_i\|_\infty \sqrt{|\text{supp } \boldsymbol{u}_i|} \\ &\le \quad \alpha \sqrt{\frac{1}{2}s - r} \\ &= \quad \frac{1}{\sqrt{2}} \alpha \sqrt{s}. \qquad (2.14) \end{aligned}$$

By (2.12) and (2.13), we have

$$\begin{aligned} &|T_2| + |\text{supp } \boldsymbol{u}_i| \le \frac{1}{2}s, \\ &|T_1| + |T_2| + |\text{supp } \boldsymbol{u}_i| \le s \qquad (2.15) \end{aligned}$$

and by the assumption (2.6) and (2.12),

$$|h_j^{T_1}| \ge \alpha \ge \|\boldsymbol{u}_i\|_\infty \ge |u_k^{\text{supp } \boldsymbol{u}_i}|$$

for each $j \in T_1$ and $k \in \text{supp } \boldsymbol{u}_i$, which implies that

$$\|\boldsymbol{h}_{T_2} + \boldsymbol{u}_i\|_2 \le \|\boldsymbol{h}_{T_1}\|_2. \qquad (2.16)$$

Hence, it follows from (2.11), (2.15), (2.16) and Lemma 2.2 that

$$\begin{aligned} |\langle A\boldsymbol{h}_{T_1}, A(\boldsymbol{h}_{T_2} + \boldsymbol{h}_{T_3})\rangle| \quad &\le \quad \sum_{i=1}^{N} \lambda_i \, |\langle A\boldsymbol{h}_{T_1}, A(\boldsymbol{h}_{T_2} + \boldsymbol{u}_i)\rangle| \\ &\le \quad \delta_s \sum_{i=1}^{N} \lambda_i \|\boldsymbol{h}_{T_1}\|_2 \|\boldsymbol{h}_{T_2} + \boldsymbol{u}_i\|_2 \\ &\le \quad \delta_s \|\boldsymbol{h}_{T_1}\|_2^2. \qquad (2.17) \end{aligned}$$

Since A obeys the RIP of order s, it follows from (2.4) and (2.17) that

$$\begin{aligned} (1 - \delta_s)\|\boldsymbol{h}_{T_1}\|_2^2 \quad &= \quad \|A\boldsymbol{h}_{T_1}\|_2^2 \\ &= \quad \langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}\rangle - \langle A\boldsymbol{h}_{T_1}, A(\boldsymbol{h}_{T_2} + \boldsymbol{h}_{T_3})\rangle \\ &\le \quad \sqrt{1 + \delta_{\frac{s}{2}}} \|\boldsymbol{h}_{T_1}\|_2 \|A\boldsymbol{h}\|_2 + |\langle A\boldsymbol{h}_{T_1}, A(\boldsymbol{h}_{T_2} + \boldsymbol{h}_{T_3})\rangle| \\ &\le \quad 2\sqrt{1 + \delta_{\frac{s}{2}}} \|\boldsymbol{h}_{T_1}\|_2 \varepsilon + \delta_s \|\boldsymbol{h}_{T_1}\|_2^2, \end{aligned}$$

which implies by the assumption: $\delta_s < \frac{1}{2}$ that

$$\|\boldsymbol{h}_{T_1}\|_2 \le \frac{2\sqrt{1 + \delta_{\frac{s}{2}}}}{1 - 2\delta_s} \varepsilon. \qquad (2.18)$$

Using (2.16) and (2.18), we can show that

$$\begin{aligned}\|\boldsymbol{x}-\boldsymbol{x}^{\star}\|_2 &\leq \|\boldsymbol{h}_{T_1}\|_2+\|\boldsymbol{h}_{T_2}+\boldsymbol{h}_{T_3}\|_2\\ &\leq 2\|\boldsymbol{h}_{T_1}\|_2\\ &\leq \frac{4\sqrt{1+\delta_{\frac{1}{2}s}}}{1-2\delta_s}\varepsilon. \end{aligned}\tag{2.19}$$

Suppose that (2.6) does not hold. Then $T_2=\emptyset$ and $T_3=T_1^c$, and so we have

$$\|\boldsymbol{h}_{T_3}\|_\infty \leq \alpha,$$
$$\|\boldsymbol{h}_{T_3}\|_1 = \frac{1}{2}\alpha s = \alpha\left(\frac{1}{2}s\right).$$

By Lemma 2.1, we have

$$\boldsymbol{h}_{T_3}=\sum_{i=1}^{N}\lambda_i \boldsymbol{u}_i,$$

where

$$\begin{aligned}\mathrm{supp}\ \boldsymbol{u}_i &\subset T_3,\\ |\mathrm{supp}\ \boldsymbol{u}_i| &\leq \frac{1}{2}s,\\ \|\boldsymbol{u}_i\|_\infty &\leq \alpha.\end{aligned}$$

Hence, it follows that

$$|T_1|+|\mathrm{supp}\ \boldsymbol{u}_i|\leq s,$$
$$\|\boldsymbol{u}_i\|_2 \leq \frac{1}{\sqrt{2}}\alpha\sqrt{s},$$

which implies

$$\begin{aligned}|\langle A\boldsymbol{h}_{T_1},A\boldsymbol{h}_{T_3}\rangle| &\leq \sum_{i=1}^{N}\lambda_i\,|\langle A\boldsymbol{h}_{T_1},A\boldsymbol{u}_i\rangle|\\ &\leq \delta_s\|\boldsymbol{h}_{T_1}\|_2\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}_i\|_2\\ &\leq \frac{1}{\sqrt{2}}\delta_s\|\boldsymbol{h}_{T_1}\|_2\alpha\sqrt{s}.\end{aligned}\tag{2.20}$$

By Lemma 2.2, we have

$$\begin{aligned}\alpha\sqrt{s} &= \frac{2}{\sqrt{s}}\|\boldsymbol{h}_{T_1^c}\|_1\\ &\leq \frac{2}{\sqrt{s}}\left(\|\boldsymbol{h}_{T_1}\|_1+2\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1\right)\\ &\leq \sqrt{2}\|\boldsymbol{h}_{T_1}\|_2+\frac{4}{\sqrt{s}}\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1,\end{aligned}\tag{2.21}$$

which implies by (2.20) that

$$|\langle A\boldsymbol{h}_{T_1},A\boldsymbol{h}_{T_3}\rangle| \leq \delta_s\|\boldsymbol{h}_{T_1}\|_2^2+\frac{2\sqrt{2}}{\sqrt{s}}\|\boldsymbol{h}_{T_1}\|_2\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1.\tag{2.22}$$

Since A obeys the RIP of order s, it follows from (2.4) and (2.22) that

$$\begin{aligned}(1-\delta_s)\|\boldsymbol{h}_{T_1}\|_2^2 &\leq \|A\boldsymbol{h}_{T_1}\|_2^2 \\ &= \langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}\rangle - \langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_3}\rangle \\ &\leq 2\sqrt{1+\delta_{\frac{s}{2}}}\|\boldsymbol{h}_{T_1}\|_2\varepsilon \\ &\quad +\delta_s\|\boldsymbol{h}_{T_1}\|_2^2 + \frac{2\sqrt{2}}{\sqrt{s}}\|\boldsymbol{h}_{T_1}\|_2\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1, \end{aligned} \tag{2.23}$$

which implies by the assumption: $\delta_s < \frac{1}{2}$ that

$$\|\boldsymbol{h}_{T_1}\|_2 \leq \frac{2\sqrt{1+\delta_{\frac{s}{2}}}}{1-2\delta_s}\varepsilon + \frac{2\sqrt{2}}{(1-2\delta_s)\sqrt{s}}\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1. \tag{2.24}$$

By (2.21), we have

$$\begin{aligned}\|\boldsymbol{h}_{T_3}\|_2 &\leq \sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}_i\|_2 \\ &\leq \frac{1}{\sqrt{2}}\alpha\sqrt{s} \\ &\leq \|\boldsymbol{h}_{T_1}\|_2 + \frac{2\sqrt{2}}{\sqrt{s}}\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1, \end{aligned}$$

which implies by (2.23) that

$$\begin{aligned}\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 &\leq \|\boldsymbol{h}_{T_1}\|_2 + \|\boldsymbol{h}_{T_3}\|_2 \\ &\leq 2\|\boldsymbol{h}_{T_1}\|_2 + \frac{2\sqrt{2}}{\sqrt{s}}\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1 \\ &\leq \frac{4\sqrt{1+\delta_{\frac{s}{2}}}}{1-2\delta_s}\varepsilon + \frac{2\sqrt{2}(3-2\delta_s)}{\sqrt{s}(1-2\delta_s)}\|\boldsymbol{x}-\boldsymbol{x}_{\frac{s}{2}}\|_1. \end{aligned} \tag{2.25}$$

This complete the proof.

2.3 Bound for $\delta_{s'}$ ($s' < s$)

Considering other decompositions $\{T_1, T_2, T_3\}$ of $\{1, 2, \cdots, n\}$, we have the following results:

Theorem 2.2. Assume that A obeys the RIP of order $\frac{3}{4}s$. Then the following (i) and (ii) hold:
(i) If

$$\frac{s}{2}\min\{|h_i|;\ i \in T_1\} \geq 2\|\boldsymbol{h}_{T_1^c}\|_1, \tag{2.26}$$

where $T_1 = \{1, 2, \cdots, \frac{1}{2}s\}$ and if $\delta_{\frac{3}{4}s} < 2-\sqrt{2} \approx 0.586$, then

$$\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 \leq \frac{2(2+\sqrt{2})\sqrt{1+\delta_{\frac{s}{2}}}}{2-(2+\sqrt{2})\delta_{\frac{3}{4}s}}\varepsilon. \tag{2.27}$$

(ii) If (2.26) does not hold and $\delta_{\frac{3}{4}s} < \sqrt{2} - 1 \approx 0.414$, then

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \quad \leq \quad & \frac{4}{\sqrt{s}(1 - (\sqrt{2}+1)\delta_{\frac{3}{4}s})}\|\boldsymbol{x} - \boldsymbol{x}_{\frac{s}{2}}\|_1 \\ & + \frac{2(\sqrt{2}+1)\sqrt{1+\delta_{\frac{s}{2}}}}{1-(\sqrt{2}+1)\delta_{\frac{3}{4}s}}\varepsilon. \end{aligned} \tag{2.28}$$

Proof. We consider a decomposition $\{T_2, T_3\}$ of T_1^c as follows:

$$\begin{aligned} T_2 &= \{i \in T_1^c;\ \ |h_i| > 2\alpha\}, \\ T_3 &= \{i \in T_1^c;\ \ |h_i| \leq 2\alpha\}, \end{aligned}$$

where $\alpha = \frac{\|\boldsymbol{h}_{T_1^c}\|_1}{s/2}$. Then, Theorem 2.2 is shown similarly to the proof of Theorem 2.1. We have omit the detailed proof.

We put

$$\alpha = \frac{\|\boldsymbol{h}_{T_1^c}\|_1}{3s/5},$$

and

$$\begin{aligned} T_1 &= \{1, 2, \cdots, \frac{3}{5}s\}, \\ T_2 &= \{i \in T_1^c;\ \ |h_i| > \frac{5}{3}\alpha\}, \\ T_3 &= \{i \in T_1^c;\ \ |h_i| \leq \frac{5}{3}\alpha\}. \end{aligned}$$

Then we can similarly show the following

Theorem 2.3. Assume that A obeys the RIP of order $\frac{24}{25}s$. Then we have the following (i) and (ii):
(i) If

$$\frac{3s}{5}\min\{|h_i|;\ \ i \in T_1\} \geq \frac{5}{3}\|\boldsymbol{h}_{T_1^c}\|_1 \tag{2.29}$$

and if $\delta_{\frac{24}{25}s} < \frac{5-\sqrt{15}}{2} \approx 0.564$, then

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq \frac{2(5+\sqrt{15})\sqrt{1+\delta_{\frac{3}{5}s}}}{5-(5+\sqrt{15})\delta_{\frac{24}{25}s}}\varepsilon. \tag{2.30}$$

(ii) If (2.29) does not hold and $\delta_{\frac{24}{25}s} < \frac{\sqrt{15}-3}{2} \approx 0.436$, then

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \quad \leq \quad & \frac{10}{\sqrt{s}(3 - (3+\sqrt{15})\delta_{\frac{24}{25}s})}\|\boldsymbol{x} - \boldsymbol{x}_{\frac{3}{5}s}\|_1 \\ & + \frac{2(3+\sqrt{15})\sqrt{1+\delta_{\frac{3}{5}s}}}{3-(3+\sqrt{15})\delta_{\frac{24}{25}s}}\varepsilon. \end{aligned} \tag{2.31}$$

3 Conclusion

In this paper, we propose the theorem 2.1, 2.2, 2.3 by using the T. Cai and A. Zhang idea and H. Inoue idea, and prepare some notions and lemmas to prove our main theorems, and introduce new bounds of δ_s and $\delta_{s'}$ $(s' < s)$. In more details, in Theorem 2.1 it is shown under the assumption that A obeys the RIP of order s and $\delta_s < \frac{1}{2}$, and if (1.5) holds, then in case of noiseless $\boldsymbol{x}$ is completely recovered as $\boldsymbol{x}^\star$ and if (1.5) does not hold, then every $\frac{s}{2}$-sparse vector $\boldsymbol{x}$ is completely recovered as $\boldsymbol{x}^\star$. This result shows that the sufficient condition for δ_s can be substantially improved, but the condition for the sparsity becomes worse. On the other hand, any sufficient conditions for $\delta_{s'}$ $(s' < s)$ have never been given. By changing a subset T_1 of T_0 or the condition (1.5), we shall similarly give sufficient conditions for $\delta_{s'}$ $(s' < s)$ in Theorem 2.2 and Theorem 2.3.

I believe that there are some applications of our theorems in other field. However, the RIP requires a bounded condition number for all submatrices built by selecting s arbitrary columns and the spectral norm of a matrix is generally difficult to calculate. Therefore, it seems useful to weaken the condition of RIP. In [9], E.J. Candès and Y. Plan have introduced the notion of weak RIP which is a generalization of RIP. In a recent paper [10], [11], H. Inoue has focused on this notion and evaluate the solution of CS under the assumption of only the weak RIP without the probability, and obtain almost the same results as for the case of the RIP. Thus it seems that the notion of weak RIP is useful in case that we have some information about the data and it seems better to analyze data using the weak RIP because it is much easier to construct matrices obeying the weak RIP than matrices obeying the RIP. Furthermore, H. Inoue has proposed the RIPless theory and the method of an unknown signal recovery in CS [12]. There are main benefits for considering the RIPless theory. First, we do not suppose that a matrix satisfies the condition of RIP. Moreover, we do not suppose the condition of sparsity. Practically, it is very difficult to know the condition of RIP and the sufficient condition of isometry constants. Likewise, we can not know the sparsity of $\boldsymbol{x}$. Second, the assessments of various cases lead to developments for signal analysis or other analysis.

We suggest that if we can apply the main results in this paper to the weak RIP and RIPless theory, it is possible to apply our theorem to other field. By using our results, the application in practical is the research task from now on.

Acknowledgment

The author is grateful to both anonymous referees for their careful reading of the paper and suggestions for improvement.

Competing Interests

The author declares that no competing interests exist.

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