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Generalized Sturm-Liouville Problems and Chebychev Collocation Method

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Abstract

In this paper, we present an algorithm for approximating the eigenvalues of Sturm-Liouville problems with parameter-dependent boundary conditions. The algorithm is based on the Chebychev method. A few examples shall be presented to illustrate the proposed method and a comparison made with the regularized sampling method. It is shown that the Chebychev method yields better results.

Keywords: Eigenvalues, Sturm-Liouville, Chebychev polynomials.

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1 Introduction

The study of many physical phenomena, such as the vibration of strings, the interaction of atomic particles, or the earths free oscillations yields Sturm-Liouville (SL) eigenvalue problems. The general form of Sturm-Liouville problems that concerns this paper is

$$\sum_{j=0}^{2} P_j(x)\, u^{(j)}(x) = \lambda^2\, q(x)\, u(x), \qquad x \in J = [0,1] \tag{1.1}$$

subject to the boundary conditions

$$\begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) & b_{11}(\lambda) & b_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & b_{21}(\lambda) & b_{22}(\lambda) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \\ u(1) \\ u'(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.2}$$

where a_{ij} and b_{ij} are functions of λ and $q(x)$, $u(x)$ and $P_j(x)$, are analytic functions . It will always be assumed that (1.1) possesses an unique solution $u \in C^n(J)$. The values of λ for which the

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boundary value problem has nontrivial solution are called eigenvalues of (1.1). A nontrivial solution corresponding to an eigenvalue is called an eigenfunction. Theorems that list conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed in [1].

Recently, Sturm-Liouville problems were treated in [2] using Tau method, in [3] using spectral method, in [4,5] using Chebyshev collocation method, in [6] using variational iteration method, in [7] using differential quadrature (DQ) method, in [8] using the Adomian decomposition method, in [9] using Haar wavelets, in [10] using boundary value methods, in [11] using the homotopy perturbation method, in [12] using Regularized sampling method, in [13,14] using the differential transformation, in [15] using finite-difference methods, in [16] using an iterative method, in [17] using mapped barycentric Chebyshev differentiation matrix method and in [18,19,20] using the sinc-collocation method.

In recent years, a lot of attention has been devoted to the study of the Chebychev method to investigate various scientific models. It is possible to solve linear differential equations [21], non-linear [22,23], integral equations [24], second and fourth-order elliptic equations [25], nonlinear system of second-order boundary value problems [26], fourth-order Sturm-Liouville problems [27] Troesch's problem [28], linear partial differential equations [29], nonlinear optimal control problems [30] and integro differential equations [31,32] by using this method. The convergence analysis of the proposed method studied by Changqing in [33].

To our knowledge there is no study on the Chebychev applications to Sturm-Liouville problems with parameter-dependent boundary conditions.

The paper is organized into five sections. Section II contains notations, definitions and some results of Chebchev polynomials. Section III is devoted to the numerical computation of the eigenvalues problem of (1.1)-(1.2). Section IV is devoted to give some examples exhibiting the technique. Finally, Section V provides conclusions of the study.

2 Some Properties of Chebychev Polynomials

The well known Chebyshev polynomials are defined on the interval $[-1, 1]$ and the following definitions are necessary for this step [34].

Definition 2.1. Chebyshev polynomial of degree n is defined as

$$T_n(x) = \cos(n \arccos(x)), \qquad n = 0, 1, \ldots, \qquad x \in [-1, 1]$$

or, in a more instructive form,

$$T_n(x) = \cos n\,\theta, \qquad x = \cos\,\theta, \qquad \theta \in [0, \pi]$$

One sees at once that, on $[-1, 1]$,

1 T_n takes its maximal value with alternating signs $(n+1)$ times:

$$\|T_n\| = 1, \qquad T_n(x_k) = (-1)^k,$$
$$x_k = \cos\left(\frac{\pi\,k}{n}\right), \quad k = 0, 1, \ldots, n \tag{2.1}$$

2 T_n has n distinct zeros:

$$T_n(t_k) = 0, \quad t_k = \cos\left(\frac{(2\,k-1)\,\pi}{2\,n}\right), \quad k = 1, 2, \ldots, n.$$

Lemma 2.1. *Chebyshev polynomials T_n satisfy the recurrence relation*

$$T_0(x) = 1, \qquad T_1(x) = x,$$
$$T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x), \qquad n \geq 1.$$

In particular, T_n is indeed an algebraic polynomial of degree n with the leading coefficient 2^{n-1}. Also, orthogonality

$$\begin{aligned} \int_{-1}^{1} T_m(x)\,T_n(x)\,\frac{d\,x}{\sqrt{1-x^2}} &= \quad 0, \qquad n \neq m \\ \int_{-1}^{1} T_n^2(x)\,\frac{d\,x}{\sqrt{1-x^2}} &= \begin{cases} \frac{\pi}{2}, & n > 0; \\ \pi, & n = 0. \end{cases} \end{aligned} \tag{2.2}$$

In this paper we use orthonormal Chebychev polynomials, noting property (2.2).

Theorem 2.2. *On the interval $[-1, 1]$, among all polynomials of degree n with leading coefficient $a_n = 1$, the Chebychev polynomial $\frac{1}{2^{n-1}} T_n$ deviates least from zero, i.e.,*

$$\textit{inf}_{(a_i)} \|x^n + a_{n-1}\,x^{n-1} + \ldots + a_0\| = \frac{1}{2^{n-1}}\,\|T_n\|.$$

We are concerned with the approximate solution by means of the Chebychev polynomials in the form

$$u(x) = {\sum_{r=0}^{N}}{}' a_r\,T_r(x), \qquad -1 \le x \le 1 \tag{2.3}$$

where $T_r(x)$ denotes Chebychev polynomials of the first kind of degree r, a_r are unknown Chebychev coefficients and N is chosen any positive integer such that $N \ge 2$. Let us assume that the first and second derivatives of the function $u(x)$ have truncated Chebychev series expansion of the form

$$u^{(k)}(x) = {\sum_{r=0}^{N}}{}' a_r^{(k)}\,T_r(x), \qquad k = 1, 2. \tag{2.4}$$

Then the solution expressed by (2.3) and its derivatives can be written in the matrix forms

$$\mathbf{A}^{(k)} = 2^k\,\mathbf{M}^k\,\mathbf{A}$$

$$\mathbf{U}^{(k)} = 2^k\,\mathbf{T}\mathbf{M}^k\,\mathbf{A}$$

where

$$\mathbf{T} = \begin{pmatrix} T_0(x_0) & T_1(x_0) & \ldots & T_N(x_0) \\ T_0(x_1) & T_1(x_1) & \ldots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & \ldots & T_N(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ T_0(x_N) & T_1(x_N) & \ldots & T_N(x_N) \end{pmatrix},$$

$$\mathbf{U}^{(k)} = \begin{pmatrix} u^{(k)}(x_0) \\ u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_N) \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix},$$

$$\mathbf{A} = \left[\frac{a_0}{2}, a_1, \ldots, a_N\right]^{\tau}$$

and

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \ldots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \ldots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \ldots & m_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}_{(N+1)\times(N+1)}. \tag{2.5}$$

where

$$m_1 = \frac{N}{2}, \qquad m_2 = 0, \qquad m_3 = N \qquad \text{if } N \text{ is odd}$$
$$m_1 = 0, \qquad m_2 = N, \qquad m_3 = 0 \qquad \text{if } N \text{ is even}$$

The method can be developed for the problem defined in the domain $[a, b]$.

Definition 2.2. On the interval $[a, b]$, the shifted Chebychev polynomial is given by

$$T_n^*(x) = T_n(y), \qquad y = \frac{2}{b-a}\left(x - \frac{a+b}{2}\right).$$

Notice that its leading coefficient is equal to $2^{n-1}\left(\frac{2}{b-a}\right)^n$.

To obtain the solution in terms of shifted Chebychev polynomials $T_r^*(x)$ in the form

$$u(x) = \sum_{r=0}^{N} a_r^* \, T_r^*(x), \qquad a \leq x \leq b,$$

where $T_r^*(x) = T_r\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right)$.

It is followed the previous procedure using the collocation points defined by

$$x_i = \frac{b-a}{2}\left[\left(\frac{a+b}{b-a}\right) + \cos\left(\frac{i\,\pi}{N}\right)\right], \qquad i = 0, 1, \ldots, N, \tag{2.6}$$

and the relation

$$A^{*(k)} = \left(\frac{4}{b-a}\right)^k \mathbf{M}^k \, \mathbf{A}^*, \qquad k = 0, 1, 2.$$

where

$$\mathbf{A} = \left[\frac{a_0^*}{2}, a_1^*, \ldots, a_N^*\right]^\tau$$

It is easily obtained $T = T^*$, because of the properties of Chebychev polynomials.

3 The Description of Chebychev Scheme

We assume that $u(x)$, the solution of (1.1), is approximated by the finite expansion of Chebychev basis functions

$$u(x) = \sum_{r=0}^{N}{}' a_r^* \, T_r^*(x), \qquad 0 \leq x \leq 1 \tag{3.1}$$

where $T_r^*(x) = T_r(2\,x - 1)$ presents the shifted Chebychev polynomials of the first kind of degree r and a_r^* for $r = 0, 1, \ldots, N$ are the undetermined Chebyshev coefficients and the Chebychev collocation points in $[0, 1]$ are

$$x_i = \frac{1}{2}\left[1 + \cos\left(\frac{i\,\pi}{N}\right)\right], \qquad i = 0, 1, \ldots, N. \tag{3.2}$$

And its derivatives have truncated Chebychev series expansion of the form

$$u^{(k)}(x) = \sum_{r=0}^{N} (a_r^*)^k \; T_r^*(x), \qquad k = 0, 1, 2. \tag{3.3}$$

Using (3.1) and (3.3) and substituting $x = x_k$ in (3.2) and applying the collocation to it, we eventually obtain the following theorem.

Theorem 3.1. *If the assumed approximate solution of the problem (1.1) is (3.1), then the discrete Chebychev system is given by*

$$\sum_{j=0}^{2} P_j(x_i)\, u^{(j)}(x_i) = \lambda^2\, q(x_i)\, u(x_i) \tag{3.4}$$

The fundamental matrix equation for(3.4) is

$$\mathbf{W}\,\mathbf{A}^* = 0 \tag{3.5}$$

where

$$\mathbf{W} = \sum_{j=0}^{2} 4^j\, \mathbf{P}_j\, \mathbf{T}^*\, \mathbf{M}^j - \mathbf{C}\,\mathbf{T}^*,$$

and

$$\mathbf{P}_j = \begin{bmatrix} p_j(x_0) & 0 & \dots & 0 \\ 0 & p_j(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_j(x_N) \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \lambda^2\, q(x_0) & 0 & \dots & 0 \\ 0 & \lambda^2\, q(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^2\, q(x_N) \end{bmatrix}$$

The boundary condition is derived from equation (1.2) and matrices for conditions are

$$a_{11}(\lambda)\,\mathbf{T}^*(0)\,\mathbf{A}^* + a_{12}(\lambda)\,\mathbf{T}^*(0)\,\mathbf{M}\,\mathbf{A}^* + b_{11}(\lambda)\,\mathbf{T}^*(1)\,\mathbf{A}^* + 4\,b_{12}(\lambda)\,\mathbf{T}^*(1)\,\mathbf{M}\,\mathbf{A}^* = 0 \tag{3.6}$$

$$a_{21}(\lambda)\,\mathbf{T}^*(0)\,\mathbf{A}^* + a_{22}(\lambda)\,\mathbf{T}^*(0)\,\mathbf{M}\,\mathbf{A}^* + b_{21}(\lambda)\,\mathbf{T}^*(1)\,\mathbf{A}^* + 4\,b_{22}(\lambda)\,\mathbf{T}^*(1)\,\mathbf{M}\,\mathbf{A}^* = 0. \tag{3.7}$$

Consequently, replacing two rows of the augmented matrix by the equation (3.5), we have

$$\widetilde{\mathbf{W}}\,\mathbf{A}^* = 0. \tag{3.8}$$

This set of equations has a non-trivial solution only if the determinant of the coefficients matrix vanishes. This gives a function of and the roots of this function are eigenvalues of the problem.

4 Examples and comparisons

In this section, we present a few examples to illustrate our method. In the following, λ_{exact} denotes the exact eigenvalue, λ_{Cheby} denotes the approximate eigenvalue computed by the method of section 3. We use the absolute error which is defined as

$$E_{Cheby} = \left|\lambda_{\text{exact}} - \lambda_{\text{Cheby}}\right|$$

Example 1: [12] We considered the following example

$$-u'' = \lambda^2\, u(x), \qquad 0 < x < 1$$

subject to the boundary conditions

$$u(0) + \left(\lambda^2 - 4\,\pi^2\right)\, u'(0) = 0$$
$$u(1) - \lambda^2\, u'(1) = 0$$

The exact characteristic function is

$$\beta_{exact} = \left(1 + 4\,\pi^2\,\lambda^4 - \lambda^6\right)\,\frac{\sin\,\lambda}{\lambda} - \left(2\,\lambda^2 - 4\,\pi^2\right)\,\cos\,\lambda$$

where zero is not an eigenvalue. The computed eigenvalues together with the "exact" ones are displayed in **Table 1**.

Table 1 The first three eigenvalues of λ_{exact} and λ_{Cheby} for Example 1 at $N = 16$.

λ_k	λ_{exact}	λ_{Cheby}
1	9.7308865782130820	9.730886578213059
2	88.763316252589763	88.76331847516812
3	157.88411043863472	157.8842052283192

Maximum absolute error are tabulated in **Table 2** for Chebychev method together with the analogous results of (12) .

Table 2 Comparison of maximum absolute errors for Example 1

λ_k	$\|E_{\text{Cheby}}\|$	Results of (12)
1	1.9539E-14	0.1554E-06
2	2.2225E-06	4.8758E-07
3	9.4789E-05	3.6142E-05

Example 2: [12] Consider the Sturm-Liouville problem

$$-u(x)'' = \lambda^2\,u(x), \qquad 0 < x < 1$$

subject to the boundary conditions

$$\begin{aligned} u(0) - 2\,u'(0) &= 0 \\ (1+\lambda)\;u(1) + (1-\lambda^2)\,u'(1) &= 0 \end{aligned}$$

The exact characteristic function is

$$\beta_{exact} = \left(2\,\cos\,\lambda + \frac{\sin\,\lambda}{\lambda}\right) + (1-\lambda)\,[-2\,\lambda\,\sin\,\lambda + \cos\,\lambda]$$

where 1 is not an eigenvalue. The computed eigenvalues λ together with the "exact" ones are displayed in **Table 3**.

Table 3 The first three eigenvalues of λ_{exact} and λ_{Cheby} for Example 2 at $N = 16$.

λ_k	λ_{exact}	λ_{Cheby}
1	0.929679054283188	0.929679054283189
2	9.9387434140	9.938743414040601
3	11.2738742105212	11.27387421054431

Maximum absolute error are tabulated in **Table 4** for Chebychev method together with the analogous results of [12].

Table 4 Comparison of maximum absolute errors for Example 2

λ_k	$\Vert E_{\text{Cheby}}\Vert$	Results of (12)
1	9.9920E-016	0.1554E-06
2	4.0602E-011	4.8758E-07
3	2.3119E-011	3.6142E-05

Example 3: [12] Consider the Sturm-Liouville problem

$$-u''(x) + e^{i\,2\,x}\,u(x) = \lambda^2\,u(x)$$

subject to the boundary conditions

$$\begin{aligned} u(0) + \lambda\,u(1) &= 0 \\ u'(0) &= 0 \end{aligned}$$

The exact characteristic function is

$$B_{exact} = \text{det}\begin{pmatrix} J_{\sqrt{\lambda}}(1) + \sqrt{\lambda}\,J_{\sqrt{\lambda}}(e^{i}) & J_{-\sqrt{\lambda}}(1) + \sqrt{\lambda}\,J_{-\sqrt{\lambda}}(e^{i}) \\ \frac{1}{2}J_{-\sqrt{\lambda}-1}(1) - \frac{1}{2}\,J_{\sqrt{\lambda}+1}(1) & \frac{1}{2}J_{\sqrt{\lambda}-1}(1) - \frac{1}{2}\,J_{-\sqrt{\lambda}+1}(1) \end{pmatrix}$$

where J_a and J_{-a} are the Bessel functions of the first kind of order a. **Table 5** lists the first three eigenvalues of Example 3 at $N = 10$.

Table 5 The first three eigenvalues in Example 3

λ_k	λ_{exact}	λ_{Cheby}
1	4.96854309+0.3906545 i	4.96854305+0.3906547 i
2	20.6027103+0.7502325 i	20.6027193+0.7502221 i
3	64.1403824+0.6842283 i	64.1382692+0.6839719 i

Example 4: [12] Consider the Sturm-Liouville problem

$$-u''(x) + e^{x}\,u(x) = \lambda^2\,u(x)$$

subject to the boundary conditions

$$\begin{aligned} u(0) &= 0 \\ -\lambda\,\sin(\lambda)u(1) + \cos(\lambda)\,u'(1) &= 0 \end{aligned}$$

Table 6 lists the first three eigenvalues of Example 4 at $N = 14$.

Table 6 The first three eigenvalues in Example 4

λ_k	λ_{exact}	λ_{Cheby}
1	0.92906202857	0.92906202844
2	6.74788117825	6.74788117868
3	16.1245477258	16.1245477689

5 Conclusion

The Chebychev technique is applied to solve Sturm-Liouville problems with parameter dependent boundary condition. We have presented a few examples to illustrate the method and compared the computed eigenvalues with the exact ones obtained as the zeros of the exact characteristic functions. Numerical results are compared to those obtained by the regularized sampling method [12] to illustrate the effectiveness of the proposed method. It is shown that the Chebychev technique is very promising in this problem. The results of example 3 clearly indicate that our method are accurate even when the coefficients p_i are complex-valued function satisfying $p_i \in L^1_{Ioc}(0, 1)$.

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Competing Interests

The authors declare that no competing interests exist.

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