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Gracefulness of H-Super Subdivision of Y-Tree

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Authors' contributions

This work was carried out in collaboration between all authors. Author KT designed the study. Authors EE and SB wrote the protocol and wrote the first draft of the manuscript and managed literature searches. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, we introduce a new method of construction called H-super subdivision of graphs and prove that the H-super subdivision of Y-tree Y_{n+1} $(n \geq 2)$ is graceful and there by answering the open question posed in 2009 by Arumugam and others [1]. Also we prove that the H-super subdivision of Y-tree is odd and even graceful.

Keywords: H-super subdivision; Y-tree; graceful; odd graceful; even graceful.

1 INTRODUCTION

The most interesting area of research in graph theory is graph labeling. It was introduced by Rosa in 1967 [2]. Rosa called a function f, a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2 \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edges are $\{1, 2, \ldots, q\}$ which are distinct. Later on this β-valuation was renamed as graceful labeling by Golomb [3]. The concept of odd

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graceful labeling was introduced by Gnanajothi [4] and it is defined as an injection $f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ such that when each edge xy is assigned a label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. A graph which admits an odd graceful labeling is called an odd graceful graph. The even graceful labeling is defined as an injection $f : V(G) \to \{0, 1, 2, \ldots, 2q\}$ such that, when each edge xy is assigned a label $|f(x) - f(y)|$, the resulting edge labels are $\{2, 4, 6, \ldots, 2q\}$. A graph which admits an even graceful labeling is called an even graceful graph [5].

Sethuraman and Selvaraju [6] have introduced the concept of super subdivision of graphs. A graph G' is said to be super subdivision of G if G' is obtained from G by replacing each edge e_i by a complete bipartite graph $K_{2,m}$ for some m in such a way that the ends of e_i are merged with the two vertices of the 2-vertices part of $K_{2,m}$ after removing the edge e_i from G. They proved that arbitrary super subdivision of paths and cycles are graceful [6]. They conjuctured that “Are there any graphs different from paths whose arbitrary super subdivision are gracedul?” Barrientos proved this conjecture by proving that every Y-tree is graceful [7]. A Y-tree Y_{n+1} $(n \geq 2)$ is a graph obtained from the path P_n by appending an edge to a vertex of the path P_n adjacent to an end vertex [8]. It is proved that arbitrary super subdivision of stars, grid graphs and cyclic snakes are graceful [9, 10]. In [1], Arumugam et al. proposed the open problem that “Are there graphs apart from $K_{2,m}$ which can be used for edge replacement in defining the super subdivision that will admit graceful labeling or α-valuation?”. In this paper we answer his question by introducing a new method of construction called H-super subdivision of graphs and prove that H-super subdivision of Y-tree admits graceful labeling. Also we prove that the H-super subdivision of Y-tree is odd and even graceful. Here we consider simple finite, connected and undirected graphs. For all terminologies and notations one may refer to Harary [11] and for graph labeling as in [8].

2 MAIN RESULTS

Definition 2.1. The H-graph is a tree on 6 vertices in which exactly two vertices of degree 3. we consider a H-graph obtained by adding an edge between even degree vertices of two paths P_2 and P_2' each of length two.

We now introduce the H-supersubdivision of a graph in the following definition.

Definition 2.2. Let G be a (p, q) graph. A graph obtained from G by replacing each edge e_i by a H-graph in such a way that the ends of e_i are merged with a pendent vertex in P_2 and a pendent vertex in P_2' is called H-super subdivision of G and it is denoted by $HSS(G)$. Thus $HSS(G)$ has $p + 4q$ vertices and $5q$ edges.

Structure of $HSS(Y_{n+1})$

Let Y_{n+1} be a Y-tree $(n \geq 2)$ with $n + 2$ vertices and $n + 1$ edges. Let the vertices of Y_{n+1} be $v_1, v_2, \ldots, v_{n+1}, u$. The $HSS(Y_{n+1})$ is constructed from Y_{n+1} by replacing each edge by the H-graph.

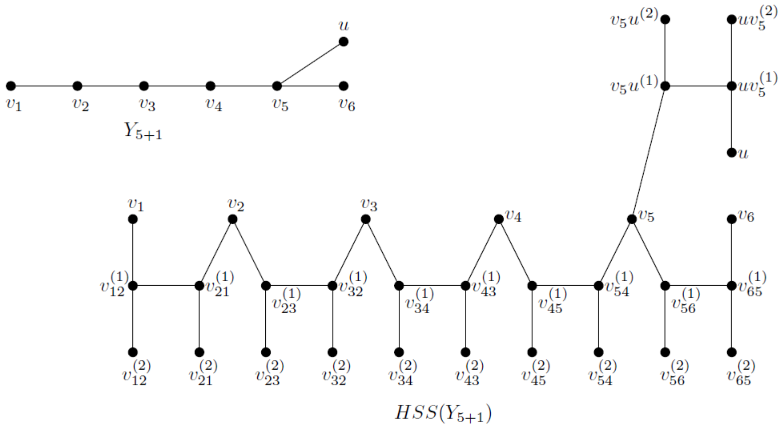
The vertex and edge sets of $HSS(Y_{n+1})$ are as follows

$$V(HSS(Y_{n+1})) = \Big\{\{u, v_nu^{(1)}, v_nu^{(2)}, uv_n^{(1)}, uv_n^{(2)}, v_{n+1}\} \\ \cup\{v_i \cup v^{(1)}_{i(i+1)} \cup v^{(2)}_{i(i+1)} \cup v^{(1)}_{(i+1)i} \cup v^{(2)}_{(i+1)i} | 1 \leq i \leq n\}\Big\}$$

and

$E(HSS(Y_{n+1})) = E_1 \cup E_2$ where
$E_1 = \{v_n(v_nu^{(1)}), (v_nu^{(1)})(v_nu^{(2)}), (v_nu^{(1)})(uv_n^{(1)}), (uv_n^{(1)})(uv_n^{(2)}), (uv_n^{(1)})(u)\}$,
$E_2 = \{v_iv^{(1)}_{i(i+1)}, v^{(1)}_{i(i+1)}v^{(2)}_{i(i+1)}, v^{(1)}_{i(i+1)}v^{(1)}_{(i+1)i}, v^{(1)}_{(i+1)i}v^{(2)}_{(i+1)i}, v^{(1)}_{(i+1)i}v_{(i+1)} | 1 \leq i \leq n\}$
Clearly this $HSS(Y_{n+1})$ has $5n + 6$ vertices and $5n + 5$ edges.

Example 2.1.



Algorithm 2.1.

Procedure: Graceful labeling of $HSS(Y_{n+1})$, $n \geq 2$

Input: $HSS(Y_{n+1})$ graph

$V \leftarrow \{u, uv_n^{(1)}, uv_n^{(2)}, v_n u^{(1)}, v_n u^{(2)},$ $v_{n+1}, v_i, v_{i(i+1)}^{(1)}, v_{i(i+1)}^{(2)},$ $v_{(i+1)i}^{(1)}, v_{(i+1)i}^{(2)} | 1 \leq i \leq n\}$

// assignment of labels to the vertices //

for $i = 1$ to n do

{

if $i \equiv 0 \pmod 2$ do

{

$v_i \leftarrow 5\left(n - \frac{i}{2}\right) + 8;$

$v_{i(i+1)}^{(1)} \leftarrow \frac{5i-4}{2};$

$v_{(i+1)i}^{(1)} \leftarrow 5\left((n+1) - \frac{i}{2}\right) + 1;$

$v_{i(i+1)}^{(2)} \leftarrow 5\left((n+1) - \frac{i}{2}\right) + 2;$

$v_{(i+1)i}^{(2)} \leftarrow \frac{5i-2}{2};$

}

else

{

$v_i \leftarrow 5\left(\frac{i-1}{2}\right);$

$v_{i(i+1)}^{(1)} \leftarrow 5\left((n+1) - \frac{(i-1)}{2}\right);$

$v_{(i+1)i}^{(1)} \leftarrow \frac{5i-1}{2};$

$v_{i(i+1)}^{(2)} \leftarrow \frac{5i-3}{2};$

$v_{(i+1)i}^{(2)} \leftarrow 5\left((n+1) - \frac{(i-1)}{2}\right) - 1;$

}

end if

}

end for

if $n \equiv 0 \pmod 2$ do

$u \leftarrow \frac{5n+2}{2};$

$uv_n^{(1)} \leftarrow \frac{5n+10}{2};$

$uv_n^{(2)} \leftarrow \frac{5n+4}{2};$

$v_n u^{(1)} \leftarrow \frac{5n+6}{2};$

$v_n u^{(2)} \leftarrow \frac{5n+8}{2};$

$v_{n+1} \leftarrow \frac{5n}{2};$

else

$u \leftarrow \frac{5n+1}{2};$

$uv_n^{(1)} \leftarrow \frac{5n+9}{2};$

$uv_n^{(2)} \leftarrow \frac{5n+3}{2};$

$v_n u^{(1)} \leftarrow \frac{5n+5}{2};$

$v_n u^{(2)} \leftarrow \frac{5n+7}{2};$

$v_{n+1} \leftarrow \frac{5(n-1)}{2} + 8;$

end if

end procedure

Output: The vertex labaled $HSS(Y_{n+1})$.

Theorem 2.1. *The H-super subdivision of Y-tree is graceful.*

Proof. Let $HSS(Y_{n+1})$ be the H-super subdivision of a Y-tree Y_{n+1} which has $5n+6$ vertices and $5n+5$ edges as given in the above structure. Label the vertices of $HSS(Y_{n+1})$ by defining a function $f : V \to \{0, 1, 2, \ldots, 5n+5\}$ as given in the algorithm 2.1. Clearly the vertices of $HSS(Y_{n+1})$ have distinct labels. Define an induced function $f^* : E \to \{1, 2, \ldots, 5n+5\}$ as $f^*(uv) = |f(u) - f(v)|$ for every $u, v \in V$. Using this induced function the edge labels of E_1 are calculated as follows:

For all n,
$f^*(v_n(v_n u^{(1)})) = 5$
$f^*((v_n u^{(1)})(v_n u^{(2)})) = 1$
$f^*((v_n u^{(1)})(u v_n^{(1)})) = 2$
$f^*((u v_n^{(1)})(u v_n^{(2)})) = 3$
$f^*((u v_n^{(1)})u) = 4$

The edge labels of E_2 are calculated as follows:
For $1 \leq i \leq n$

Case (i): $i \equiv 0 \pmod 2$

$$f^*(v_i v^{(1)}_{i(i+1)}) = |f(v_i) - f(v^{(1)}_{i(i+1)})| = |5(n-i+2)|$$

$$f^*(v^{(1)}_{i(i+1)} v^{(2)}_{i(i+1)}) = |f(v^{(1)}_{i(i+1)}) - f(v^{(2)}_{i(i+1)})| = |5(i-n)-9|$$

$$f^*(v^{(1)}_{i(i+1)} v^{(1)}_{(i+1)i}) = |f(v^{(1)}_{i(i+1)}) - f(v^{(1)}_{(i+1)i})| = |5(i-n)-8|$$

$$f^*(v^{(1)}_{(i+1)i} v^{(2)}_{(i+1)i}) = |f(v^{(1)}_{(i+1)i}) - f(v^{(2)}_{(i+1)i})| = |5(n-i)+7|$$

$$f^*(v^{(1)}_{(i+1)i} v_{(i+1)}) = |f(v^{(1)}_{(i+1)i}) - f(v_{(i+1)})| = |5(n-i)+6|$$

Case (ii): $i \equiv 1 \pmod 2$

$$f^*(v_i v^{(1)}_{i(i+1)}) = |f(v_i) - f(v^{(1)}_{i(i+1)})| = |5(i-n-2)|$$

$$f^*(v^{(1)}_{i(i+1)} v^{(2)}_{i(i+1)}) = |f(v^{(1)}_{i(i+1)}) - f(v^{(2)}_{i(i+1)})| = |5(n-i)+9|$$

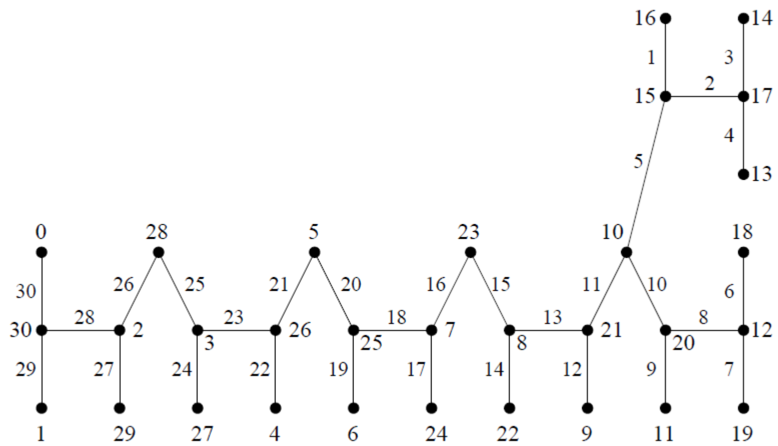
$$f^*(v^{(1)}_{i(i+1)} v^{(1)}_{(i+1)i}) = |f(v^{(1)}_{i(i+1)}) - f(v^{(1)}_{(i+1)i})| = |5(n-i)+8|$$

$$f^*(v^{(1)}_{(i+1)i} v^{(2)}_{(i+1)i}) = |f(v^{(1)}_{(i+1)i}) - f(v^{(2)}_{(i+1)i})| = |5(i-n)-7|$$

$$f^*(v^{(1)}_{(i+1)i} v_{(i+1)}) = |f(v^{(1)}_{(i+1)i}) - f(v_{(i+1)})| = |5(i-n)-6|$$

Thus the edges of $HSS(Y_{n+1})$ have distinct labels. Hence $HSS(Y_{n+1})$ is graceful. □

Example 2.2. *Graceful labeling of H-super subdivision of Y_{5+1}.*



Proposition 2.1. *The H-super subdivision of path graphs is graceful.*

Proof. Since $HSS(P_n)$ is a caterpillar, it is graceful. □

3 ODD AND EVEN GRACEFUL LABELINGS OF $HSS(Y_{n+1})$

Algorithm 3.1
Procedure: Odd graceful labeling of $HSS(Y_{n+1})$
Input: $HSS(Y_{n+1})$ graph
$V \leftarrow \{u, uv_n^{(1)}, uv_n^{(2)}, v_n u^{(1)}, v_n u^{(2)}, v_{n+1}\} \cup \{v_i, v_{i(i+1)}^{(1)}, v_{i(i+1)}^{(2)},$
$v_{(i+1)i}^{(1)}, v_{(i+1)i}^{(2)} / 1 \leq i \leq n\}$
// assignment of labels to the vertices //
for $i = 1$ to n do
{
if $i \equiv 0$ (mod 2)
{
$v_i \leftarrow 5(2n - i + 3);$
$v_{i(i+1)}^{(1)} \leftarrow 5i - 4;$
$v_{i(i+1)}^{(2)} \leftarrow 5(2n - i + 2) + 3;$
$v_{(i+1)i}^{(1)} \leftarrow 5(2n - i + 2) + 1;$
$v_{(i+1)i}^{(2)} \leftarrow 5i - 2;$
}
else
{
$v_i \leftarrow 5(i - 1);$
$v_{i(i+1)}^{(1)} \leftarrow 5(2n - i + 2) + 4;$
$v_{i(i+1)}^{(2)} \leftarrow 5i - 3;$
$v_{(i+1)i}^{(1)} \leftarrow 5i - 1;$
$v_{(i+1)i}^{(2)} \leftarrow 5(2n - i + 2) + 2;$
}
end if
}
end for
if $n \equiv 0$ (mod 2)
{
$v_{n+1} \leftarrow 5n;$
$u \leftarrow 5n + 2;$
$uv_n^{(1)} \leftarrow 5n + 9;$
$uv_n^{(2)} \leftarrow 5n + 4;$
$v_n u^{(1)} \leftarrow 5n + 6;$
$v_n u^{(2)} \leftarrow 5n + 7;$
}
else
{
$v_{n+1} \leftarrow 5(n + 2);$
$u \leftarrow 5n + 8;$
$uv_n^{(1)} \leftarrow 5n + 1;$
$uv_n^{(2)} \leftarrow 5n + 6;$

$v_n u^{(1)} \leftarrow 5n + 4;$
$v_n u^{(2)} \leftarrow 5n + 3;$
}
end if
end procedure
Output: The vertex labeled $HSS(Y_{n+1})$

Theorem 3.1. *The H-super subdivision of Y-tree is odd graceful.*

Proof. Let $HSS(Y_{n+1})$ be the H-super subdivision of a Y-tree Y_{n+1} which has $5n + 6$ vertices and $5n + 5$ edges.

Label the vertices of $HSS(Y_{n+1})$ by defining an injective function $f : V \to \{0, 1, 2, \ldots, 2q - 1\}$ as given in the algorithm 3.1. Clearly the vertices of $HSS(Y_{n+1})$ have distinct labels. Define an induced function $f^* : E \to \{1, 3, \ldots, 2q - 1\}$ as $f^*(uv) = |f(u) - f(v)|$ for every $u, v \in V$. Using this induced function the edge labels of E_1 are calculated as follows.

For all n,
$f^*(v_n(v_n u^{(1)})) = 9$
$f^*((v_n u^{(1)})(v_n u^{(2)})) = 1$
$f^*((v_n u^{(1)})(uv_n^{(1)})) = 3$
$f^*((uv_n^{(1)})(uv_n^{(2)})) = 5$
$f^*((uv_n^{(1)})u) = 7$

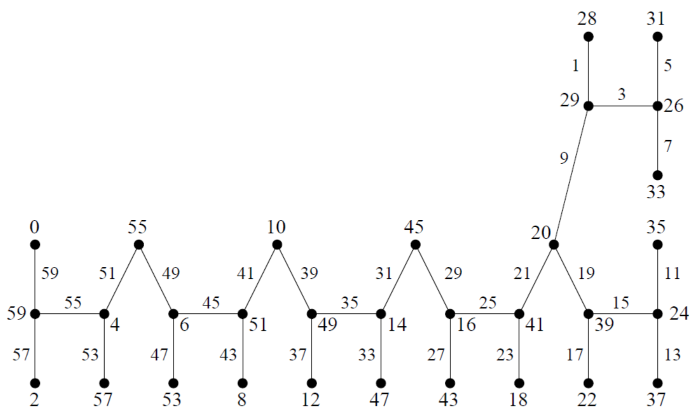
The edge labels of E_2 are calculated as follows.

For $1 \leq i \leq n$,
Case (i): $i \equiv 0 \pmod 2$
$f^*(v_i(v_{i(i+1)}^{(1)})) = |10(n - i) + 19|$
$f^*(v_{i(i+1)}^{(1)} v_{i(i+1)}^{(2)})) = |10(i - n) - 17|$
$f^*(v_{i(i+1)}^{(1)} v_{(i+1)1}^{(1)})) = |10(i - n) - 15|$
$f^*(v_{(i+1)i}^{(1)} v_{(i+1)i}^{(2)})) = |10(n - i) + 13|$
$f^*(v_{(i+1)i}^{(1)} v_{i+1}) = |10(n - i) + 11|$

Case (ii): $i \equiv 1 \pmod 2$
$f^*(v_i(v_{i(i+1)}^{(1)})) = |10(i - n) - 19|$
$f^*(v_{i(i+1)}^{(1)} v_{i(i+1)}^{(2)})) = |10(n - i) + 17|$
$f^*(v_{i(i+1)}^{(1)} v_{(i+1)1}^{(1)})) = |10(n - i) + 15|$
$f^*(v_{(i+1)i}^{(1)} v_{(i+1)i}^{(2)})) = |10(i - n) - 13|$
$f^*(v_{(i+1)i}^{(1)} v_{i+1}) = |10(i - n) - 11|$

Thus the edges of $HSS(Y_{n+1})$ have distinct odd labels.
Hence $HSS(Y_{n+1})$ is odd graceful. □

Example 3.1. *Odd graceful labeling of H-super subdivision of Y_{5+1}.*



Algorithm 3.2
Procedure: Even graceful labeling of $HSS(Y_{n+1})$ graph
Input: $HSS(Y_{n+1})$ graph

$V \leftarrow \{u, uv_n^{(1)}, uv_n^{(2)}, v_nu^{(1)}, v_nu^{(2)}, v_{n+1}\} \cup \{v_i, v_{i(i+1)}^{(1)}, v_{i(i+1)}^{(2)},$
$\quad v_{(i+1)i}^{(1)}, v_{(i+1)i}^{(2)} / 1 \le i \le n\}$

// assignment of labels to the vertices //
for $i = 1$ to n do
{
  if $i \equiv 0 \pmod 2$
  {
    $v_i \leftarrow 5(2n - i + 3) + 1;$
    $v_{i(i+1)}^{(1)} \leftarrow 5i - 4;$
    $v_{i(i+1)}^{(2)} \leftarrow 5(2n - i + 3) - 1;$
    $v_{(i+1)i}^{(1)} \leftarrow 5(2n - i + 2) + 2;$
    $v_{(i+1)i}^{(2)} \leftarrow 5i - 2;$
  }
  else
  {
    $v_i \leftarrow 5(i - 1);$
    $v_{i(i+1)}^{(1)} \leftarrow 5(2n - i + 3);$
    $v_{i(i+1)}^{(2)} \leftarrow 5i - 3;$
    $v_{(i+1)i}^{(1)} \leftarrow 5i - 1;$
    $v_{(i+1)i}^{(2)} \leftarrow 5(2n - i + 2) + 3;$
  }
  end if
}
end for
if $n \equiv 0 \pmod 2$
{
  $v_{n+1} \leftarrow 5n;$
  $u \leftarrow 5(n + 2);$
  $uv_n^{(1)} \leftarrow 5n + 2;$
  $uv_n^{(2)} \leftarrow 5n + 8;$
  $v_nu^{(1)} \leftarrow 5n + 6;$

$v_n u^{(2)} \leftarrow 5n+4;$
}
else
{
$v_{n+1} \leftarrow 5(n+2)+1;$
$u \leftarrow 5n+1;$
$uv_n^{(1)} \leftarrow 5n+9;$
$uv_n^{(2)} \leftarrow 5n+3;$
$v_n u^{(1)} \leftarrow 5n+5;$
$v_n u^{(2)} \leftarrow 5n+7;$
}
end if
end procedure
Output: The vertex labeled $HSS(Y_{n+1})$

Theorem 3.2. *The H-super subdivision of Y-tree is even graceful.*

Proof. Let $HSS(Y_{n+1})$ be the H-super subdivision of a Y-tree Y_{n+1} which has $5n+6$ vertices and $5n+5$ edges.

Label the vertices of $HSS(Y_{n+1})$ by defining an injective function $f : V \to \{0,1,2,\ldots,2q\}$ as given in the algorithm 3.2.

Clearly the vertices of $HSS(Y_{n+1})$ have distinct labels.

Define an induced function $f^* : E \to \{2,4,\ldots,2q\}$ as $f^*(uv) = |f(u) - f(v)|$ for every $u, v \in V$.

Using this induced function the edge labels of E_1 are calculated as follows.

For all n,
$f^*(v_n(v_n u^{(1)})) = 10$
$f^*((v_n u^{(1)})(v_n u^{(2)})) = 2$
$f^*((v_n u^{(1)})(uv_n^{(1)})) = 4$
$f^*((uv_n^{(1)})(uv_n^{(2)})) = 6$
$f^*((uv_n^{(1)})u) = 8$

The edge labels of E_2 are calculated as follows.

For $1 \le i \le n$,
Case (i): $i \equiv 0 \pmod 2$
$f^*(v_i(v^{(1)}_{i(i+1)})) = |10(n-i+2)|$
$f^*(v^{(1)}_{i(i+1)} v^{(2)}_{i(i+1)})) = |10(i-n)-18|$
$f^*(v^{(1)}_{i(i+1)} v^{(1)}_{(i+1)1})) = |10(i-n)-16|$
$f^*(v^{(1)}_{(i+1)i} v^{(2)}_{(i+1)i})) = |10(n-i)+14|$
$f^*(v^{(1)}_{(i+1)i} v_{i+1}) = |10(n-i)+12|$

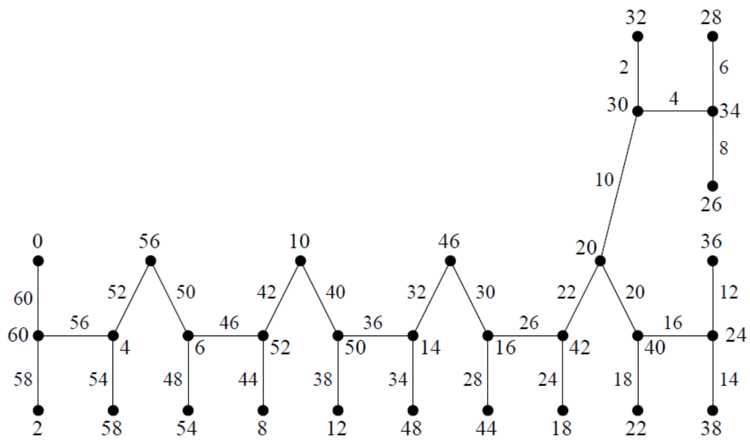
Case (ii): $i \equiv 1 \pmod 2$
$f^*(v_i(v^{(1)}_{i(i+1)})) = |10(i-n-2)|$
$f^*(v^{(1)}_{i(i+1)} v^{(2)}_{i(i+1)})) = |10(n-i)+18|$

$f^*(v^{(1)}_{i(i+1)}v^{(1)}_{(i+1)1})) = |10(n-i)+16|$
$f^*(v^{(1)}_{(i+1)i}v^{(2)}_{(i+1)i})) = |10(i-n)-14|$
$f^*(v^{(1)}_{(i+1)i}v_{i+1}) = |10(i-n)-12|$

Thus the edges of $HSS(Y_{n+1})$ have distinct even labels.

Hence $HSS(Y_{n+1})$ is even graceful. □

Example 3.2. *Even graceful labeling of H-super subdivision of Y_{5+1}.*



4 CONCLUSION

We defined H-super subdivision of a graph and proved that H-super subdivision Y-tree admits graceful, odd and even graceful labelings.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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