Archives of Current Research International

19(1): 1-4, 2019; Article no.ACRI.52346
ISSN: 2454-7077

Amply Cofinitely Essential Supplemented Modules

Berna Koşar[1] and Celil Nebiyev[2*]

[1] *Faculty of Health Sciences, Üsküdar University, Turkey.*
[2] *Department of Mathematics, Ondokuz Mayıs University, Kurupelit-Atakum/Samsun/Türkİye, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ACRI/2019/v19i130146
Editor(s):
(1) Dr. Sanjeev Kumar, Professor, Department of Physics, Medical Physics Research Laboratory, D.A.V College, India.
Reviewers:
(1) Naresh Kumar, Mewar University, India.
(2) Pasupuleti Venkata Siva Kumar, Vallurupalli Nageswara Rao Vignana Jyothi Institute of Engineering & Technology, India.
(3) Ricardo Lus Lima Vitria, Universidade Federal do Esprito Santo, Brazil.
Complete Peer review History: http://www.sdiarticle4.com/review-history/52346

Received 16 August 2019
Accepted 30 October 2019
Published 05 November 2019

Original Research Article

ABSTRACT

Let M be an $R-$module. If every cofinite essential submodule of M has ample supplements in M then M is called an amply cofinitely essential supplemented module. In this work some properties of these modules are investigated. Let M be a projective and cofinitely essential supplemented $R-$module. Then every $M-$generated $R-$module is amply cofinitely essential supplemented.

Keywords: Essential submodules; cofinite submodules; small submodules; supplemented modules.

2010 Mathematics Subject Classification: 16D10; 16D70.

**Corresponding author: E-mail: cnebiyev@omu.edu.tr;*

1 INTRODUCTION

In this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. It is denoted a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M with $M = N + L$, then N is called a *small* (or *superfluous)* submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule of M and denoted by $N \trianglelefteq M$ if $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. N is called a *cofinite* submodule of M if M/N is finitely generated. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (or briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement $V'$with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. M is said to be *amply essential supplemented* (or briefly, *amply e-supplemented*) if every essential submodule of M has ample supplements in M. If every cofinite submodule of M has ample supplements in M, then M is called an *amply cofinitely supplemented* module. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$.

Some properties of supplemented modules are in [1] and [2]. Some information about cofinitely supplemented modules are in [3]. The definition of e-supplemented module and some properties of this module are in [4]. Some properties of amply essential supplemented modules are in [5].

Definition 1.1 Let M be an $R-$module. If every cofinite essential submodule of M has a supplement in M, then M is called a cofinitely essential supplemented (or briefly, cofinitely e-supplemented) module. (See [6])

Definition 1.2 Let M be an $R-$module and $X \leq M$. If X is a supplement of a cofinite essential submodule of M, then X is called a ce-supplement submodule in M. (See [6])

Lemma 1.3 Let M be an cofinitely essential supplemented module. Then every $M-$generated $R-$module is cofinitely essential supplemented. (See [6])

2 AMPLY COFINITELY ESSENTIAL SUPPLEMENTED MODULES

Definition 2.1 Let M be an $R-$module. If every cofinite essential submodule of M has ample supplements in M, then M is called an amply cofinitely essential supplemented (or briefly, amply ce-supplemented) module.

Clearly, every amply ce-supplemented module is cofinitely essential supplemented.

Proposition 2.2 Let M be an amply ce-supplemented module. Then $M/RadM$ have no proper cofinite essential submodules.

Proof. Since M is amply ce-supplemented, then M is cofinitely essential supplemented. Then by [6, Proposition 2.4], $M/RadM$ have no proper cofinite essential submodules. □

Lemma 2.3 Let M be an amply ce-supplemented module. Then every factor module of M is amply ce-supplemented.

Proof. Let $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K}$ be a cofinite essential submodule of $\frac{M}{K}$ and $\frac{M}{K} = \frac{U}{K} + \frac{V}{K}$. Then U is a cofinite essential submodule of M and $M = U + V$. Since M is amply ce-supplemented, U has a supplement X in M with $X \leq V$. Since $K \leq U$, by [2, 41.1 (7)], $\frac{X+K}{K}$ is a supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Moreover, $\frac{X+K}{K} \leq \frac{V}{K}$. Hence $\frac{M}{K}$ is amply ce-supplemented. □

Corollary 2.4 Every homomorphic image of an amply ce-supplemented module is amply ce-supplemented.

Proof. Clear from Lemma 2.3. □

Definition 2.5 Let M be an $R-$module. If every proper cofinite essential submodules of M is small in M or M has not proper cofinite essential submodules, then M is called an ce-hollow module.

Lemma 2.6 Every ce-hollow module is amply ce-supplemented.

Proof. Clear from definitions. □

Corollary 2.7 Every ce-hollow module is cofinitely essential supplemented.

Proof. Clear from Lemma 2.6, since every amply ce-supplemented module is cofinitely essential supplemented. □

Lemma 2.8 If M is a π-projective and cofinitely essential supplemented module, then M is an amply ce-supplemented module.

Proof. Let U be a cofinite essential submodule of M, $M = U + V$ and X be a supplement of U in M. Since M is π -projective and $M = U + V$, there exists an R -module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im\,(1 - f) \subset U$. So, we have $M = f\,(M) + (1 - f)\,(M) = f\,(U) + f\,(X) + U = U + f\,(X)$. Suppose that $a \in U \cap f\,(X)$. Since $a \in f\,(X)$, then there exists $x \in X$ such that $a = f\,(x)$. Since $a = f\,(x) = f\,(x) - x + x = x - (1 - f)\,(x)$ and $(1 - f)\,(x) \in U$, we have $x = a + (1 - f)\,(x) \in U$. Thus $x \in U \cap X$ and so, $a = f\,(x) \in f\,(U \cap X)$. Therefore we have $U \cap f\,(X) \leq f\,(U \cap X) \ll f\,(X)$. This means that $f\,(X)$ is a supplement of U in M. Moreover, $f\,(X) \subset V$. Therefore M is amply ce-supplemented. □

Corollary 2.9 If M is a projective and cofinitely essential supplemented module, then M is an amply ce-supplemented module.

Proof. Clear from Lemma 2.8. □

Lemma 2.10 Let M be a π-projective $R-$module. If every cofinite essential submodule of M is β^* equivalent to a ce-supplement submodule in M, then M is amply ce-supplemented. The definition of β^* $relation\ and\ some\ properties\ of\ this\ relation\ are\ in$ [7].

Proof. By [6, Lemma 2.13], M is cofinitely essential supplemented. Since M is π-projective, by Lemma 2, M is amply ce-supplemented. □

Corollary 2.11 Let M be a projective $R-$module. If every cofinite essential submodule of M is β^* equivalent to a ce-supplement submodule in M, then M is amply ce-supplemented.

Proof. Clear from Lemma 2.10. □

Lemma 2.12 Let Λ be any index set and $\{M_\lambda\}_\Lambda$ be a family of projective $R-$modules. If M_λ is cofinitely essential supplemented for every $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply ce-supplemented.

Proof. Since M_λ is cofinitely essential supplemented for every $\lambda \in \Lambda$, by [6, Lemma 2.7], $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is cofinitely essential supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by [2, 18.1], $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective. Since $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective and cofinitely essential supplemented, by Corollary 2.9,, $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply ce-supplemented. □

Corollary 2.13 Let M be a projective and cofinitely essential supplemented $R-$module. Then $M^{(\Lambda)}$ is amply ce-supplemented for every index set Λ.

Proof. Clear from Lemma 2.12. □

Corollary 2.14 Let M be a projective $R-$module. If M is cofinitely essential supplemented, then every $M-$generated $R-$module is amply ce-supplemented.

Proof. Let N be a $M-$generated $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is projective and cofinitely essential supplemented, by Corollary 2, $M^{(\Lambda)}$ is amply ce-supplemented. Then by Corollary 2, N is amply ce-supplemented. □

Lemma 2.15 Let M be an $R-$module. If every submodule of M is cofinitely essential supplemented, then M is amply ce-supplemented.

Proof. Let U be a cofinite essential submodule of M and $M = U + V$ with $V \leq M$. Since $U \trianglelefteq M$, $U \cap V \trianglelefteq V$. Since $\frac{M}{U} = \frac{U+V}{U} \cong \frac{V}{U \cap V}$ and M/U finitely generated, $\frac{V}{U \cap V}$ also finitely generated and $U \cap V$ is a cofinite submodule of V. By hypothesis, V is cofinitely essential supplemented. Then $U \cap V$ has a supplement X in V. By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll X$. Then $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll X$. Hence M is amply ce-supplemented. □

Corollary 2.16 Let M be an $R-$module. If every submodule of M is cofinitely essential supplemented, then every submodule of M is amply ce-supplemented.

Proof. Clear from Lemma 2.15. □

Lemma 2.17 Let R be any ring. The following assertions are equivalent.

(i) Every $R-$module is cofinitely essential supplemented.

(ii) Every $R-$module is amply ce-supplemented.

Proof. $(i) \Longrightarrow (ii)$ Let A be any $R-$module and $K \leq A$. By hypothesis, K is cofinitely essential supplemented. hence every submodule of A is cofinitely essential supplemented and by Lemma 2, A is amply ce-supplemented. $(ii) \Longrightarrow (i)$ Clear, since every amply ce-supplemented module is cofinitely essential supplemented. □

Proposition 2.18 Let R be a ring. The following assertions are equivalent.

$(i)_R R$ is essential supplemented.

$(ii)_R R$ is amply essential supplemented.

(iii) Every finitely generated $R-$module is essential supplemented.

(iv) Every finitely generated $R-$module is amply essential supplemented.

(v) Every $R-$module is cofinitely essential supplemented.

(vi) Every $R-$module is amply ce-supplemented.

Proof. $(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv)$ are in [5].

$(i) \Longleftrightarrow (v)$ See [6, Proposition 2.11].

$(v) \Longleftrightarrow (vi)$ Clear from Lemma 2. □

3 CONCLUSION

Let R *be a ring. Then* $_RR$ *is essential supplemented if and only if every* $R-$*module is amply ce-supplemented.*

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

[1] Clark J, Lomp C, Vanaja N, Wisbauer R. Lifting modules supplements and projectivity in module theory. Frontiers in Mathematics, Birkhauser, Basel; 2006.

[2] Wisbauer R. Foundations of module and ring Theory. Gordon and Breach, Philadelphia; 1991.

[3] Alizade R, Bilhan G, Smith PF. Modules whose maximal submodules have supplements. Communications in Algebra. 2001;29(6):2389-2405.

[4] Nebiyev C, Ökten HH, Pekin A. Essential supplemented modules. International Journal of Pure and Applied Mathematics. 2018;120(2):253-257.

[5] Nebiyev C, Ökten HH, Pekin A. Amply essential supplemented modules. Journal of Scientific Research & Reports. 2018;21(4):1-4.

[6] Berna Koşar, Celil Nebiyev. Cofinitely essential supplemented modules. Turkish Studies, Information Technologies & Applied Sciences. 2018;13(29):83-88.

[7] Birkenmeier GF, Mutlu FT, Nebiyev C, Sokmez N, Tercan A. Goldie*-supplemented modules. Glasgow Mathematical Journal. 2010;52A:41-52.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sdiarticle4.com/review-history/52346