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Research Article

A New Construction of Holditch Theorem for Homothetic Motions in C_p

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In this study, the planar kinematics has been studied in a generalized complex plane which is a geometric representation of the generalized complex number system. Firstly, the planar kinematic formulas with one parameter for homothetic motions in the generalized complex plane have been mentioned briefly. Then, the Steiner area formula given areas of the trajectories drawn by the points taken in a generalized complex plane have been obtained during the one-parameter planar homothetic motion. Finally, the Holditch theorem, which gives the relationship between these areas of trajectories, has been expressed for homothetic motions in a generalized complex plane. So, this theorem obtained in this study is the most general form of all Holditch theorems obtained so far.

1. Introduction

The first scientists which introduced the complex numbers, which are expressed as $x+iy$ where the imaginer unit is i ($i^2=-1$), are thought to be Italian mathematicians Cardan (1501-1576) and Bombelli (1526-1572). In 1545, Cardan published a study called "The Great Art" and defined an algebraic formula for solving cubic and quartic equations in that study. But Cardan did not consider complex numbers in detail. On the other hand, Bombelli introduced (complex) numbers with Cardan's formula 30 years after Cardan's study. Then, it has been observed that alternative number systems can be created with the help of complex numbers. The English geometrician Clifford (1845-1879) introduced hyperbolic numbers using $i^2=+1$ [1–4]. The application to mechanics of hyperbolic numbers given by Clifford has been supported by applications to non-Euclidean geometries. Moreover, the German geometrician study introduced another number system called "dual numbers" by adding another unit to complex numbers [4–6].

Cayley-Klein geometry, which contains Euclidean, Galilean, and Minkowskian geometries, was introduced for the first time by Klein and Cayley [7, 8]. Then, Yaglom considered these studies given by Klein and Cayley and divided these geometries into three (parabolic, elliptic, and hyperbolic) according to the length measure between two points on a line and the angle measure between two lines [9]. This distinction has showed nine paths by measures of angle and length. These nine plane geometries are given in Table 1.

The generalized complex number system was introduced by Yaglom [4]. The generalized complex numbers also play a role in Cayley-Klein geometry as ordinary complex numbers play a role in Euclidean geometry [4, 9]. Then, Harkin and Harkin studied the generalized complex number system taking these studies into account [10].

As a subbranch of physics, mechanics examines the motion of systems, their effects that cause motion, and the equilibrium states of systems. Mechanics divide into three parts as statics, kinematics, and dynamics. Statics, kinematics, and dynamics examine, respectively, the equilibrium states of systems, the motion of systems without adding force, and the factors that change motion. The main elements considered in kinematics are also length and time. In dynamics, there are three basic elements that are important: length, time, and mass. Thus, kinematics can be called a science between geometry and dynamics. Briefly, kinematics

Table 1: Nine Cayley-Klein geometries in the plane.

		Measure of length		
		Elliptic	Parabolic	Hyperbolic
Measure of angles	Elliptic	Elliptic geometry	Euclidean geometry	Hyperbolic geometry
	Parabolic (Euclidean)	Co-Euclidean geometry	Galilean geometry	Co-Minkowskian geometry
	Hyperbolic	Cohyperbolic geometry	Minkowskian geometry	Doubly hyperbolic geometry

examines how the geometric properties of systems change over time. The treatment of kinematics as an independent science dates back to Ampére's (1775-1836) time, who first defined this branch of science and named it. According to Ampére, "Kinematics includes in everything that can be said about the motions of systems, regardless of the forces that move the systems. It examines all the designs of the force independent motion, taking into account many calculations such as the area formed during the motion, the time required to create this area and the various connections that can be found between this area and this time" [11]. In kinematics, the one-parameter planar motion has been studied by many scientists in different spaces. Müller introduced the one-parameter planar motion in both Euclidean and complex planes [12]. The one-parameter planar motion in the Lorentzian plane was also studied by Ergin [13] and Görmez [14]. Then, Yüce and Kuruoğlu introduced the one-parameter planar motion in the hyperbolic plane [15]. Moreover, Akar and Yüce studied in the Galilean plane [16]. Holditch theorem first expressed by Holditch [17] is one of the important theorems of kinematics. The most important point of that classical Holditch theorem is that the area is independent of the selected curve. Therefore, due to the Holditch theorem being open to technical application and its independence from the curve, it has gained a lot of attention with the choices of the plane that carries the curve and has been generalized by many scientists with different methods. Steiner gave the area formula bounded by the trajectory curve drawn by any point under a one-parameter closed planar motion [18, 19]. Since the above-mentioned studies received great interest during the one-parameter closed planar motion, many of the scientists working on kinematics have generalized the Holditch theorem with various studies. The most important of these studies are [12, 20–25]. Many studies about the Holditch theorem have been also made for homothetic motions. Tutar and Kuruoğlu obtained the Steiner area formula and Holditch theorem for the one-parameter planar homothetic motions in the Euclidean plane [26]. In addition, Kuruoğlu and Yüce generalized the area formulas for planar motions and gave the corresponding formulas for homothetic motions [27–30]. This study is the most general form of all study about Holditch theorems obtained so far.

In [31], a new analytical geometry method that is symmetries in the Euclidean plane was developed to calculate the trajectories of mechatronic systems and CAD/CAM. In addition, some examples of the design of kinematic mechanisms were presented [31]. Then, in [32], a new alternative calculation method of high accuracy calculation of robot trajectory for the complex curves was proposed. On the other hand, the similarity method has found a place in science for several centuries. The basis of the study in [33] is closely linked to mathematical linguistics. This approach has led to new results in analytical geometry used in different applications in information technology. Moreover, in [33], an architectural calculation tool was proposed and the existence of symmetry in natural languages was briefly demonstrated. The Renishaw Ballbar QC20-W is designed for the diagnosis of CNC machine tools but is also used in conjunction with industrial robots. In the standard measurement situation, not all robot joints move when the measurement plane is parallel to the robot base. In this regard, in [34], the hypothesis of motion of all robot joints has been valuated when the desired circular path was placed on an inclined plane. Therefore, the hypothesis established in the first part of the experiments was confirmed by spatial analysis on a simulation model of the robot. Then, practical measurements were made evaluating the effect of individual robot joints to deform the circular path, which is shown as a pole plot [34].

The generalized complex number system was expressed as

$$C_{\mathrm{p}} = \left\{ x + iy : x, y \in R, i^2 = p \in R \right\}, \tag{1}$$

by Yaglom and Harkin [4, 9, 10]. This system involves complex $(p = -1)$, dual $(p = 0)$, and hyperbolic $(p = +1)$ number systems and also different planes for other values of $p \in R$. Considering the aforementioned studies, some kinematic studies have been carried out in the generalized complex plane obtained from this number system. Erişir et al. obtained the Steiner area formula, the polar moment of inertia, and Holditch-type theorem in C_{p} [35, 36]. In addition to that, Erişir and Güngör gave the Holditch-type theorem for nonlinear points in a generalized complex plane C_{p}, [37, 38]. Moreover, Gürses et al. gave the one-parameter planar homothetic motion in $C_J = \{x + Jy : x, y \in R, J^2 = p, p \in \{-1, 0, 1\}\} \subset C_{\mathrm{p}}$ [39].

This study is on kinematics for one parameter planar homothetic motion in a generalized complex plane which is a geometric representation of the generalized complex number system $C_{\mathrm{p}} = \{x + iy : x, y \in R, i^2 = p \in R\}$. The Steiner formula and Holditch theorem for these homothetic motions in C_p have been obtained. So, this study is the most general version of all the studies about the Holditch theorem done so far.

2. Preliminaries

The generalized complex number system consists of ordered pairs $Z = (x, y)$ or $Z = x + iy$, and this number system specifically includes ordinary, dual, and double numbers where i^2

is also written as $i^2 = (q, p)$, $(i^2 = iq + p)$, and $x, y, q, p \in R$. So, in cases where $p + q^2/4$ is negative, zero, and positive, generalized complex numbers are isomorphic to ordinary, dual, and double numbers, respectively [4, 9, 10]. In this paper, especially, $q = 0$, $-\infty < p < \infty$, and $i^2 = p \in R$ are considered. So, the generalized complex number system is reduced

$$C_p = \{x + iy : x, y \in R, i^2 = p \in R\}. \tag{2}$$

Now, the two generalized complex numbers are considered $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2 \in C_p$. So, it can be written as

$$Z_1 \pm Z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2). \tag{3}$$

Moreover, the product in this system is

$$M^p(Z_1, Z_2) = (x_1x_2 + py_1y_2) + i(x_1y_2 + x_2y_1) \tag{4}$$

[4, 10, 40].

On the other hand, if two generalized complex vectors which are position vectors of the generalized complex numbers Z_1, Z_2 are considered $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \in C_p$, the scalar product of these vectors is

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_p = \mathrm{Re}\,(M^p(z_1, \bar{z}_2)) = \mathrm{Re}\,(M^p(\bar{z}_1, z_2)) = x_1y_1 - px_2y_2 \tag{5}$$

[10]. Moreover, the p-magnitude of the generalized complex number $Z = x + iy \in C_p$ is

$$|Z|_p = \sqrt{\left|M^p(Z, \bar{Z})\right|} = \sqrt{|x^2 - py^2|}, \tag{6}$$

where "–" denotes the ordinary complex conjugation [10]. Moreover, the unit circle in C_p is characterized by the form of $|Z|_p = 1$. Thus, the unit circle in the plane C_p can be given as Figure 1 for the special cases of p.

If the special case $p < 0$ is chosen, the unit ellipses formed by $x^2 + |p|y^2 = 1$ are obtained. Moreover, the generalized complex number system $C_p(p < 0)$ equals to the elliptical complex number system. In particular, if $p = -1$ then, the unit circle in C_p corresponds to Euclidean unit circle $x^2 + y^2 = 1$ [10]. If the situation $p = 0$ is considered, $x^2 = 1$ and the unit circles are formed as $x = \pm 1$. Moreover, the system C_0 is equal to the parabolic number system and the plane in this situation corresponds to the Galilean plane [10]. Finally, considering the special situation $p > 0$, the hyperbolas are obtained by $|x^2 - py^2| = 1$ which have asymptote $y = \pm x/\sqrt{p}$. So, the number system C_p is equal to the hyperbolic complex number system. Particularly, when $p = 1$, the generalized complex plane is the Lorentzian plane [10].

Considering the above-mentioned description of circle for cases of p, the circle in C_p can be defined as follows.

Definition 1. Let the circle which has the center $M(a, b)$ and the radius r be considered. Thus, the equation of this circle is

$$\left|(x - a)^2 - p(y - b)^2\right| = r^2 \tag{7}$$

[10].

Let a number in C_p be $Z = x + iy$ which symbolize $\overrightarrow{OT}$ and Figure 2 be as follows.

So, the angle θ_p formed by inverse tangent functions can be defined as

$$\theta_p = \begin{cases} \dfrac{1}{\sqrt{|p|}} \tan^{-1}\left(\sigma\sqrt{|p|}\right), & p < 0, \\ \sigma, & p = 0, \\ \dfrac{1}{\sqrt{p}} \tan^{-1}(\sigma\sqrt{p}), & p > 0(\text{branch I, III}), \end{cases} \tag{8}$$

where $\sigma \equiv y/x$. Let the point N be the intersection point of OT with unit circle in C_p. Moreover, the orthogonal projection on the OM of the point N is the point L and the line QM is also the tangent at the point M of the unit circle (see Figure 3). Thus, p-trigonometric functions (the p-cosine ($\cos p$), p-sine ($\sin p$), and p-tangent ($\tan p$)) can be obtained by

$$\begin{gathered} \cos p\theta_p = \begin{cases} \cos\left(\theta_p\sqrt{|p|}\right), & p < 0, \\ 1, & p = 0(\text{branch I}), \\ \cos h(\theta_p\sqrt{p}), & p > 0(\text{branch I}), \end{cases} \\ \sin p\theta_p = \begin{cases} \dfrac{1}{\sqrt{|p|}} \sin\left(\theta_p\sqrt{|p|}\right), & p < 0, \\ \theta_p, & p = 0(\text{branch I}), \\ \dfrac{1}{\sqrt{p}} \sin h(\theta_p\sqrt{p}), & p > 0(\text{branch I}), \end{cases} \end{gathered} \tag{9}$$

and the ratio $QM/OM = NL/OL$ gives

$$\tan p\theta_p = \frac{\sin p\theta_p}{\cos p\theta_p}. \tag{10}$$

Thus, the Maclaurin expansions of the p-trigonometric functions on the branch I are

$$\begin{gathered} \cos p\theta_p = \sum_{n=0}^{\infty} \frac{p^n}{(2n)!}\theta_p^{2n}, \\ \sin p\theta_p = \sum_{n=0}^{\infty} \frac{p^n}{(2n+1)!}\theta_p^{2n+1}. \end{gathered} \tag{11}$$

By the help of the Maclaurin series, the generalized Euler

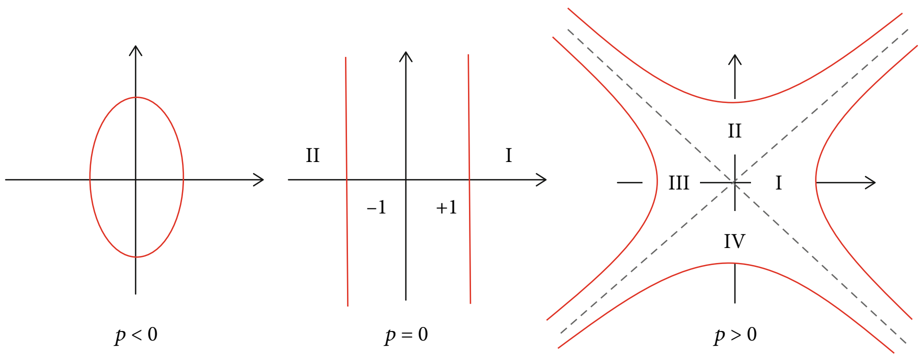

Figure 1: The unit circle in C_p.

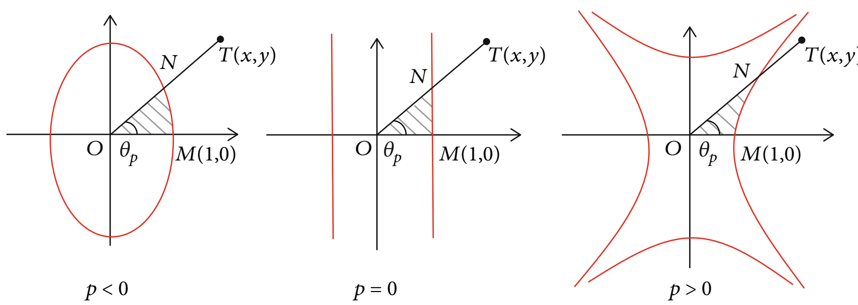

Figure 2: Elliptic, parabolic, and hyperbolic angles.

formula in C_p is

$$e^{i\theta_p} = \cos p\theta_p + i \sin p\theta_p, \tag{12}$$

where $i^2 = p$. On the other hand, the exponential forms of Z in C_p are

$$Z = r_p\left(\cos p\theta_p + i \sin p\theta_p\right) = r_p e^{i\theta_p}, \tag{13}$$

where $r_p = |Z|_p$ [10]. Moreover, the p-rotation matrix given by the help of equation (12) is

$$A\left(\theta_p\right) = \begin{bmatrix} \cos p\theta_p & p \sin p\theta_p \\ \sin p\theta_p & \cos p\theta_p \end{bmatrix} \tag{14}$$

[10].

The one parameter homothetic motions in the p-complex plane

$$C_J = \left\{x + Jy : x, y \in R, J^2 = p, p \in \{-1, 0, 1\}\right\}, \tag{15}$$

which is the subset of the generalized complex plane C_p was studied by Gürses et al. [39]. Similar to that study, the one-parameter homothetic motions in the generalized complex plane C_p have been given as follows briefly.

Let K_p, K'_p be the moving and fixed planes in C_p, respectively, and $x = x_1 + ix_2$ and $x' = x'_1 + ix'_2$ be the position vectors of a point X, and $\overrightarrow{OO'} = \mathbf{u}$. So, the equation of the one-parameter planar homothetic motion in the generalized complex plane C_p is written by

$$x' = (hx - \mathbf{u})e^{i\theta_p}, \tag{16}$$

where θ_p is the p-rotation angle of the motion K_p/K'_p, $\mathbf{u}' = -\mathbf{u}e^{i\theta_p}$, and h is the homothetic scale in C_p. So, the relative and absolute velocity vectors of X in $K_p \subset C_p$ are

$$\mathbf{V}_r{}' = \mathbf{V}_r e^{i\theta_p} = h x e^{i\theta_p}, \tag{17}$$

$$\mathbf{V}_a{}' = \mathbf{V}_a e^{i\theta_p} = \left(\dot{h} + i\dot{\theta}_p h\right) x e^{i\theta_p} - \left(\mathbf{u} + i\dot{\theta}_p \mathbf{u}\right) e^{i\theta_p} + h x e^{i\theta_p}, \tag{18}$$

respectively. Using equations (17) and (18) the guide velocity vector is

$$\mathbf{V}_f{}' = \mathbf{V}_f e^{i\theta_p} = \left(\dot{h} + i\dot{\theta}_p h\right) x e^{i\theta_p} + \mathbf{u}'. \tag{19}$$

Theorem 2. *Let K_p/K'_p be the one parameter planar homothetic motion in C_p. So, the relationship between velocity vectors is given by*

$$\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r. \tag{20}$$

There are some points that remain fixed in both the fixed plane K'_p and the moving plane K_p in C_p. These points are called pole points. Thus, let the pole points of the one-

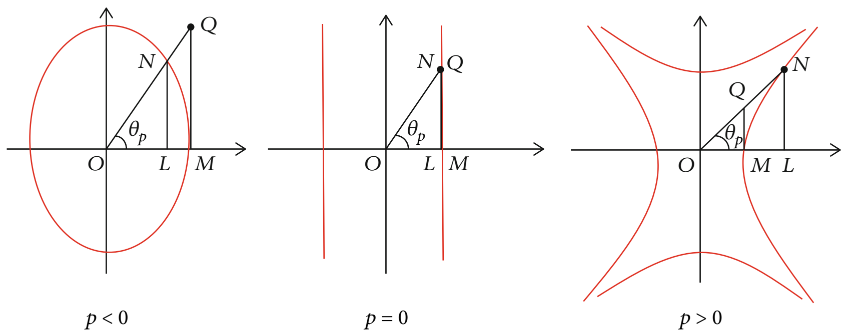

FIGURE 3: θ_p for the special cases of p.

parameter planar homothetic motions K_p/K'_p be $Q = (q_1, q_2) \in C_p$. So, the components of pole points $Q = (q_1, q_2)$ are

$$q_1 = \frac{dh(du_1 + pu_2 d\theta_p) - ph(du_2 + u_1 d\theta_p)d\theta_p}{dh^2 - ph^2 d\theta_p{}^2}, \quad q_2 = \frac{dh(du_2 + u_1 d\theta_p) - h(du_1 + pu_2 d\theta_p)d\theta_p}{dh^2 - ph^2 d\theta_p{}^2}, \tag{21}$$

where $V_f = 0$. In addition to that, the guide velocity vector of the fixed point X with respect to K_p can be written in terms of the pole points as

$$\mathbf{d}_x{}' = (dh + ihd\theta_p)(x - \mathbf{q})e^{i\theta_p}, \tag{22}$$

where $\mathbf{q}$ is the position vector of the pole point Q.

On the other hand, the following proposition, which can be proved easily, for the homothetic motions in C_p can be given.

Proposition 3. *Let two arbitrary generalized complex vectors be $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in C_p. Thus, the following equations are satisfied:*

$$(i)\ \left[\mathbf{a}e^{i\theta_p}, \mathbf{b}e^{i\theta_p}\right] = [\mathbf{a}, \mathbf{b}],$$
$$(ii)\ [\mathbf{a}, (dh + ihd\theta_p)\mathbf{b}] = [\mathbf{a}, \mathbf{b}]dh + \frac{1}{2}[\mathbf{ab} + \mathbf{ab}]hd\theta_p, \tag{23}$$

where h is the homothetic scale and

$$[\mathbf{a}, \mathbf{b}] = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1. \tag{24}$$

In this paper, the open motions restricted to time interval $[t_1, t_2]$ on branch I of C_p are considered.

3. Main Theorems and Proofs

Let $K'_p, K_p \subset C_p$ be the fixed and moving generalized complex planes, respectively, and any fixed point in K_p be $X = (x_1, x_2)$. Moreover, F_X is considered the area of trajectory drawn by the point X. So, this area is given by

$$F_X = \frac{1}{2}\int_{t_1}^{t_2} \left[\mathbf{x}', \mathbf{dx}'\right], \tag{25}$$

where the determinant product is [,] [41]. Now, considering equations (16), (22), and (23), equation (25) is equal to

$$\begin{aligned} F_X = {} & \frac{1}{2}\mathbf{x}\bar{\mathbf{x}}\int_{t_1}^{t_2} h^2 d\theta_p - \frac{1}{4}\mathbf{x}\int_{t_1}^{t_2} h^2\bar{\mathbf{q}}d\theta_p - \frac{1}{4}\bar{\mathbf{x}}\int_{t_1}^{t_2} h^2\mathbf{q}d\theta_p \\ & + \frac{1}{2}\int_{t_1}^{t_2} (u_1q_2 - u_2q_1 - hx_1q_2 + hx_2q_1 + x_1u_2 - x_2u_1)dh \\ & + \frac{1}{2}\int_{t_1}^{t_2} (u_1q_1 - pu_2q_2 - u_1x_1 + px_2u_2)hd\theta_p, \end{aligned} \tag{26}$$

where the position vector of the pole point $Q = (q_1, q_2)$ is $\mathbf{q}$ and "–" is the ordinary complex conjugate. We should note here that X is any fixed point in K_p. Particularly, if X is considered as the origin point of K_p, then, for the point $X = 0$, equation (26) is obtained that

$$F_0 = \frac{1}{2}\left[\int_{t_1}^{t_2} (u_1q_2 - u_2q_1)dh + \int_{t_1}^{t_2} (u_1q_1 - pu_2q_2)hd\theta_p\right], \tag{27}$$

where $\dot{\theta}_p \neq 0$ and $\dot{\theta}_p$ is a continuous function. So, $\dot{\theta}_p$ can be $\dot{\theta}_p < 0$ or $\dot{\theta}_p > 0$. In here, $\dot{\theta}_p$ has the same sign everywhere in the interval $[t_1, t_2]$. Now, let the mean value theorem of integral calculus for the interval $[t_1, t_2]$ be considered. So, there exists at least one point $t_0 \in [t_1, t_2]$ so that

$$\int_{t_1}^{t_2} h^2 d\theta_p = \int_{t_1}^{t_2} h^2(t)\dot{\theta}_p(t)dt = h^2(t_0)\delta_p, \tag{28}$$

where $\delta_p = \theta_p(t_2) - \theta_p(t_1)$ is the total rotation angle.

On the other hand, the Steiner point which is the center of gravity of the moving pole curve was first expressed by Steiner [18]. So, let the Steiner point in this study be represented by $S=(s_1,s_2)$. If this point is adapted to the generalized complex plane for homothetic motions, the following equations are obtained:

$$\begin{aligned}S=s_1+is_2&=\frac{1}{2ph^2(t_0)\delta_p}\left(2p\int_{t_1}^{t_2}h^2qd\theta_p-i\int_{t_1}^{t_2}hdu\right),\\2h^2(t_0)\delta_ps_1&=2\int_{t_1}^{t_2}h^2q_1d\theta_p-\int_{t_1}^{t_2}hdu_2,\\2ph^2(t_0)\delta_ps_2&=2p\int_{t_1}^{t_2}h^2q_2d\theta_p-\int_{t_1}^{t_2}hdu_1,\end{aligned}\tag{29}$$

where h is the homothetic scale in C_p. In here, it is known that for $p=-1$ homothetic motion in the complex plane, in the hyperbolic plane for $p=+1$, and in the dual plane for $p=0$ are mentioned. If $p=0$, this situation differs from other cases ($p\neq0$); instead of a Steiner point, a Steiner line forms as $s_2=s_2(\gamma(t))$. After all these calculations, equation (26) is obtained that

$$\begin{aligned}F_X=F_0&+\frac{1}{2}h^2(t_0)\delta_p\left(x_1{}^2-px_2{}^2-2x_1s_1+2px_2s_2\right)\\&+\frac{1}{2}x_1\int_{t_1}^{t_2}(-2hq_2+u_2)dh+\frac{1}{2}x_2\int_{t_1}^{t_2}(2hq_1-u_1)dh.\end{aligned}\tag{30}$$

So, if the equations $\zeta_1=1/2\int_{t_1}^{t_2}(-2hq_2+u_2)dh$ and $\zeta_2=1/2\int_{t_1}^{t_2}(2hq_1-u_1)dh$ are considered, the following theorem can be given.

Theorem 4. *For the homothetic motion K_p/K'_p in C_p, the Steiner area formula giving the area of the trajectory formed by the fixed point X is*

$$F_X=F_0+\frac{1}{2}h^2(t_0)\delta_p(\mathbf{x}\bar{\mathbf{x}}-\mathbf{x}\bar{\mathbf{s}}-\bar{\mathbf{x}}\mathbf{s})+\zeta_1x_1+\zeta_2x_2,\tag{31}$$

where $S=(s_1,s_2)$ is the Steiner point of homothetic motion and h is the homothetic scale in C_p.

Particularly, if $h=1$ is considered, equation (31) is obtained

$$F_X=F_0+\frac{1}{2}\delta_p(\mathbf{x}\bar{\mathbf{x}}-\mathbf{x}\bar{\mathbf{s}}-\bar{\mathbf{x}}\mathbf{s})\tag{32}$$

in [35].

Let F_X, the area of the trajectory drawn by the point $X=(x_1,x_2)$, be constant. So, from equation (31), the equation

$$x_1{}^2-px_2{}^2-2\left(s_1-\frac{\zeta_1}{h^2(t_0)\delta_p}\right)x_1+2\left(ps_2-\frac{\zeta_2}{h^2(t_0)\delta_p}\right)x_2+\frac{2(F_0-F_X)}{h^2(t_0)\delta_p}=0\tag{33}$$

can be written. So, the following corollaries can be obtained.

Corollary 5. *The geometric location of the points X with the same area F_X is a circle in C_p with center*

$$M=\left(s_1-\frac{\zeta_1}{h^2(t_0)\delta_p},s_2+\frac{\zeta_2}{ph^2(t_0)\delta_p}\right)\tag{34}$$

for the one-parameter planar homothetic motion in C_p.

Corollary 6. *If $h=1$, the geometric location of the points X with the same area F_X is a circle in C_p with center Steiner point $S=(s_1,s_2)$ in C_p [35].*

Now, let the Steiner area formula in equation (31) be generalized using three linear points for homothetic motions in C_p. For this, let three points X, Y, and Z in the moving plane K_p be considered that X and Y are two points and the other point Z is on XY. Moreover, let three vectors $\overrightarrow{O'X}=x'$, $\overrightarrow{O'Y}=y'$, and $\overrightarrow{O'Z}=z'$ be position vectors of these points according to K'_p. So, the relationship between these vectors is

$$z'=\alpha x'+\beta y',\ \alpha+\beta=1,\quad \alpha,\beta\in R,\tag{35}$$

where α and β are barycentric coordinates of z'. Thus, considering equation (25), the area of the trajectory drawn by Z is written by $F_Z=1/2\int_{t_1}^{t_2}[\mathbf{z}',\mathbf{dz}']$ and the area

$$\begin{aligned}F_Z=&\frac{1}{2}\alpha^2\int_{t_1}^{t_2}\left[\mathbf{x}',d\mathbf{x}'\right]+\frac{1}{2}\alpha\beta\int_{t_1}^{t_2}\left(\left[\mathbf{x}',d\mathbf{y}'\right]+\left[\mathbf{y}',d\mathbf{x}'\right]\right)\\&+\frac{1}{2}\beta^2\int_{t_1}^{t_2}\left[\mathbf{y}',d\mathbf{y}'\right]\end{aligned}\tag{36}$$

is obtained where

$$\begin{aligned}F_X&=\frac{1}{2}\int_{t_1}^{t_2}\left[\mathbf{x}',d\mathbf{x}'\right],\\F_Y&=\frac{1}{2}\int_{t_1}^{t_2}\left[\mathbf{y}',d\mathbf{y}'\right],\\F_{XY}&=\frac{1}{4}\int_{t_1}^{t_2}\left(\left[\mathbf{x}',d\mathbf{y}'\right]+\left[\mathbf{y}',d\mathbf{x}'\right]\right).\end{aligned}\tag{37}$$

So, equation (35) is written by

$$F_Z = \alpha^2 F_X + 2\alpha\beta F_{XY} + \beta^2 F_Y, \quad (38)$$

where

$$\begin{aligned} F_{XY} = F_O + \frac{1}{4}h^2(t_0)\delta_p(xy + xy - (x+y)s - (x+y)s) \\ + \frac{1}{2}\zeta_1(x_1 + y_1) + \frac{1}{2}\zeta_2(x_1 + y_1) \end{aligned} \quad (39)$$

or

$$\begin{aligned} F_{XY} = F_0 + \frac{1}{2}h^2(t_0)\delta_p(x_1y_1 - px_2y_2 - (x_1+y_1)s_1 + p(x_2+y_2)s_2) \\ + \frac{1}{2}\zeta_1(x_1 + y_1) + \frac{1}{2}\zeta_2(x_1 + y_1), \end{aligned} \quad (40)$$

where $\zeta_1 = 1/2\int_{t_1}^{t_2}(-2hq_2 + u_2)dh$ and $\zeta_2 = 1/2\int_{t_1}^{t_2}(2hq_1 - u_1)$ dh.

Let $X = Y$ be considered in equation (40). So, the equation

$$F_X = F_0 + \frac{1}{2}h^2(t_0)\delta_p(\mathbf{x}\bar{\mathbf{x}} - \mathbf{x}\bar{\mathbf{s}} - \bar{\mathbf{x}}\mathbf{s}) + \zeta_1 x_1 + \zeta_2 x_2 \quad (41)$$

is obtained. As can be seen from here, equation (41) is the same as the Steiner area formula in equation (31) for the homothetic motions in C_p. Thus, equation (40) is a more general form of the formula in equation (31). In addition, considering some calculations, the equation

$$F_X - 2F_{XY} + F_Y = \frac{1}{2}h^2(t_0)\delta_p\left({x_1}^2 - p{x_2}^2 - 2x_1y_1 + 2px_2y_2 + {y_1}^2 - p{y_2}^2\right) \quad (42)$$

is obtained. Now, let the distance between the points X and Y be d. So, using the definition of distance (6) in C_p, the distance is

$$d^2 = (x_1 - y_1)^2 - p(x_2 - y_2)^2 \quad (43)$$

for branch I of C_p. So, equation (41) can be written as

$$F_{XY} = \frac{1}{2}(F_X + F_Y) - \frac{1}{4}h^2(t_0)\delta_p d^2. \quad (44)$$

Thus, for the area of the trajectory drawn by Z, the following theorem can be given.

Theorem 7. *During the homothetic motions in C_p, the area of the trajectory drawn by Z is*

$$F_Z = \alpha F_X + \beta F_Y - \frac{1}{2}\alpha\beta h^2(t_0)\delta_p d^2, \quad (45)$$

where h is the homothetic scale, $\alpha + \beta = 1$, and F_X and F_Y are the areas formed by the points X and Y, respectively, in C_p.

During the one-parameter homothetic motions in C_p, if $F_X = F_Y$, equation (44) can be obtained that

$$F_X - F_Z = \frac{1}{2}\alpha\beta h^2(t_0)\delta_p d^2, \quad (46)$$

where $\alpha + \beta = 1$. So,

$$F_X - F_Z = \frac{1}{2}h^2(t_0)\delta_p|XZ||YZ|, \quad (47)$$

where $|XZ| = \beta d$ and $|YZ| = \alpha d$. Thus, the following main theorem can be given with the above proof.

Theorem 8. (Main Theorem). *Let two points X and Y in $K_p \subset C_p$ be fixed and the point Z be on the line XY during the homothetic motions. Moreover, when the endpoints of XY draw the same curve, the point Z on XY draws a different curve. So, the relationship between areas formed by these curves depends on the p-distances of Z to X and Y, the p-rotation angle of the homothetic motion, and homothetic scale h in C_p.*

This theorem is called Holditch theorem for the one-parameter homothetic motions in C_p. So, this theorem is the most general form of all Holditch theorems obtained so far.

4. Conclusion

Curves are very important in kinematic mechanisms. The trajectory drawn by a point or set of points (rigid body; such as a robot) along the motion creates a curve. This curve can be special curves such as a circle, ellipse, hyperbola, or a random curve formed by trajectory drawn by any point. It is important to characterize the motion to make calculations such as the area and moment of the trajectory (curve) drawn along the motion. The Holditch theorem is a theorem that expresses the area of the trajectory drawn during the motion. To be more specific, the Holditch theorem in plane geometry emphasizes that if a fixed-length chord is allowed to rotate in a convex closed curve, the position of a point on the chord x units from one end and y units from the other end, the curve drawn by this point is less than the area of the original curve πxy. This theorem was first given in 1858 by the English mathematician Hamnet Holditch. Although not emphasized by Holditch, the proof of the theorem requires the chord to be short enough that the position of the point taken is a simple closed curve. The fact that the area of trajectories expressed in the Holditch theorem is independent of the curve (circle, ellipse, etc.) makes this theorem very interesting. Thus, the Holditch theorem has been included as one of Clifford A. Pickover's 250 milestones in the history of mathematics. It should be noted again that the most important feature of the theorem is that the formula that gives the area πxy is independent of both the shape and size of the original curve, and this formula gives the same formula as the area of an ellipse with axes x and y. Until now, Holditch's theorem has been generalized to many planes and spaces. But since the generalized complex plane mentioned in this study includes hyperbolic, dual, and complex planes, and planes in other possible choices of $p \in R$, the study in this plane is a very extended study. In addition, the fact that this study is for homothetic motions adds another generalization to the study. So, this study is the most general study covering all the studies about the Holditch theorem in the plane. In future studies, areas of the trajectories formed by the curves drawn by nonlinear points in the generalized complex plane can be calculated and moment calculations can be made in this plane to contribute to engineering studies. In addition to that, this study may have presented a geometric method for calculating the areas of complex trajectories of mechatronic systems. Theoretical and practical researches are required additionally.

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The author declares no conflicts of interest.

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