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Research Article

Common Best Proximity Point Theorems for Generalized Proximal Weakly Contractive Mappings in b-Metric Space

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In this paper, common best proximity point theorems for weakly contractive mapping in b-metric spaces in the cases of nonself-mappings are proved; we introduced the notion of generalized proximal weakly contractive mappings in b-metric spaces and proved the existence and uniqueness of common best proximity point for these mappings in complete b-metric spaces. We also included some supporting examples that our finding is more generalized with the references we used.

1. Introduction

The metric fixed point theory gained impetus due to its wide range of applicability to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics, and so forth. The first fixed point theorem was given by Brouwer [1], but the credit of making concept useful and popular goes to polish mathematician, Banach [2] who proved the famous contraction mapping theorem in 1922 in the setting of metric space. This principle guarantees the existence and uniqueness of fixed point of certain self-maps of metric spaces and provides a constructive method to find those fixed points. This principle includes different directions in different spaces adopted by mathematicians for example metric spaces, G-metric spaces, partial metric spaces, and cone metric spaces.

A classical best approximation theorem was introduced by Fan [3], which states that "if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \longrightarrow B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$." Afterwards, Prolla [4], Reich [5], and Sehgal and Singh [6] have derived extensions of Fan Theorem in many directions. The common fixed point theorem insists to the authors to investigation on common best proximity point theorem for nonself-mappings. The common best proximity point theorem assures a common optimal solution at which both the real valued multiobjective functions $x \longrightarrow d(x, Sx)$ and $x \longrightarrow d(x, Tx)$ attain the global minimal value $d(A, B)$. A number of authors have improved, generalized, and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or extend best proximity results from fixed point theory (see [7–12]).

Definition 1. Let X be a nonempty set and $T : X \longrightarrow X$ a self-map. A point $x \in X$ is said to be fixed point of T if $Tx = x$.

Example 2. Let $X = \mathbb{R}$ and $T : X \longrightarrow X$ defined by $Tx = x/2$, for each $x \in X$.

$Tx = x \Rightarrow x/2 = x$, and we get $x = 0 \in X$, which is a fixed point of T.

Definition 3 (see [13]). A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous

(ii) $\psi(t) = 0$ if and only if $t = 0$

Example 4. Define $\psi : [0,\infty) \longrightarrow [0,\infty)$ by $\psi(t) = t^2/2$.

$\psi'(t) = 2t/2 = t \geq 0$, which shows ψ is nondecreasing, satisfies that $\psi(t) = 0 \Leftarrow t = 0$, and ψ is continuous.

Definition 5 (see [14]). Let X be a nonempty set, and a mapping $d : X \times X \longrightarrow [0,\infty)$ is said to be metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) > 0$ if $x \neq y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

Example 6. Let $X = \mathbb{R}$; then, $(X, |.|)$ that means $d(x, y) = |x - y|$, for all $x, y \in X$ which is a metric space.

Definition 7 (see [2]). Let (X, d) be a metric space and $T : X \longrightarrow X$ be a self-map; then, T is said to be a contraction mapping if there exists a constant $k \in [0, 1)$, such that $d(Tx, Ty) \leq kd(x, y)$, $\forall x, y \in X$.

Example 8. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$, and a mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $Tx = x/3$, $\forall x, y \in \mathbb{R}$.

Then, $d(Tx, Ty) = |Tx - Ty| = |x/3 - y/3| = |1/3(x - y)| \leq 1/3|x - y| = 1/3d(x, y)$, which implies that $d(Tx, Ty) \leq 1/3d(x, y)$ and $k = 1/3 \in [0, 1)$, for all $x, y \in X$. Therefore, T is contraction mapping.

Definition 9 (see [15]). Let (X, d) be a metric space. The mapping $T : X \longrightarrow X$ is said to be contractive mapping if

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y. \quad (1)$$

Example 10. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$, and a mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $Tx = x/2$, $\forall x, y \in \mathbb{R}$.

Then, $d(Tx, Ty) = |Tx - Ty| = |x/2 - y/2| = |1/2(x - y)| \leq 1/2|x - y| < |x - y|$, which implies that $d(Tx, Ty) < d(x, y)$ with $x \neq y$.

Therefore, T is contractive mapping.

Definition 11 (see[16]). Let (X, d) be a metric space and $T : X \longrightarrow X$, a mapping T is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \text{ for all } x, y \in X, \quad (2)$$

where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is altering function.

Remark 12. If $\phi(t) = (1 - k)t$ with $k \in [0, 1)$ and $t \in [0,\infty)$, a weak contraction reduces to a contraction.

Example 13. Let $X = [0,\infty)$ be endowed by $d(x, y) = |x - y|$, and let $T : X \longrightarrow X$ define by $Tx = x/1 + x$ for each, $x \in X$.

Define $\phi : [0,\infty) \longrightarrow [0,\infty)$ by $\phi(t) = t^2/1 + t$.

Claim: T is weakly contractive.

$\phi'(t) = 2t + t^2/(1 + t)^2 \geq 0$ which shows ϕ is nondecreasing and satisfies that $\phi(t) = 0 \Leftarrow t = 0$ and ϕ is continuous.

Then, $d(Tx, Ty) = |x/1 + x - y/1 + y| = |x - y|/(1 + x)(1 + y) \leq |x - y|/1 + |x - y| = |x - y| - |x - y|^2/1 + |x - y| = d(x, y) - \phi(d(x, y))$, for all $x, y \in X$.

So T is weakly contractive.

Definition 14 (see [17]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \longrightarrow [0,\infty)$ is said to be a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) > 0$ if $x \neq y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq sd(x, z) + sd(z, y)$

Remark 15 (see [18]). We should note that a b-metric space with $s = 1$ is a metric space. We can find several examples of b-metric spaces which are not metric spaces.

Example 16 (see [19]). Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then, $d(x, y)$ is a b-metric space with

$$s = 2^{p-1}. \quad (3)$$

Definition 17 (see [20]). Let (X, d) be a b-metric space with parameter $s \geq 1$. Then, a sequence $\{x_n\}$ in X is said to be

(i) b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$

(ii) a b-Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, for all $n, m \in \mathbb{N}$

In addition, a b-metric space is called complete if and only if each Cauchy sequence in this space is b-convergent.

Example 18. Let $X = [0,\infty)$ and $d(x, y) = (x - y)^2$; then, the space (X, d) is a complete b-metric space.

Definition 19 (see [21]). Let f and g be two self-mappings on a nonempty set X. If $w = fx = gx$, for some $x \in X$, then x is said to be the coincidence point of f and g, where w is called the point of coincidence of f and g. Let $C(f, g)$ denote the set of all coincidence points of f and g.

Definition 20 (see [21]). Let f and g be two self-mappings defined on a nonempty set X. Then, f and g are said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$, for every $x \in C(f, g)$.

Example 21.

(i) $f, g : R \longrightarrow R$ defined by $f(x) = x/3$ and $g(x) = x^2$, $x \in \mathbb{R}$. In this example, f and g have coincidence

point at $x = 0$, and $x = 1/3$ but f and g are not weakly compatible

(ii) $X = [0, 3]$ equipped with the usual metric space $d(x, y) = |x - y|$

Define $f, g : X \longrightarrow X$ by the following:

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1), \\ 3, & \text{if } x \in [1, 3], \end{cases} \qquad g(x) = \begin{cases} 3 - x, & \text{if } x \in [0, 1), \\ 3, & \text{if } x \in [1, 3]. \end{cases} \tag{4}$$

This example shows, for any $x \in [1, 3]$, $fgx = gfx$. Therefore, f and g are weakly compatible maps on $[0, 3]$.

In this study, motivated and inspired by Yan Hao and Hongyan Guan [22], we introduce the notion of generalized proximal weakly contractive mappings in b-metric spaces and prove a common best proximity point theorem for generalized proximal weakly contractive mapping defined on complete b-metric spaces.

2. Preliminaries

Definition 22 (see [23]). Let A and B be nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B), \text{for some } y \in B\}, \tag{5}$$

$$B_0 = \{y \in B : d(x, y) = d(A, B), \text{for some } x \in A\}, \tag{6}$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ is the distance between A and B.

Definition 23 (see [24]). Let A, B be nonempty subset of metric space (X, d). Given a nonself-mapping $T : A \longrightarrow B$, then an element $x^* \in A$ is called best proximity point of the mapping if

$$d(x^*, Tx^*) = d(A, B). \tag{7}$$

Definition 24 (see [25]). Let $f, g : A \longrightarrow B$ be nonself-mappings. An element $x^\star \in A$ is said to be a common best proximity point of the pair (f, g) if this condition is satisfied:

$$d(x^\star, fx^\star) = d(A, B) = d(x^\star, gx^\star). \tag{8}$$

Definition 25 (see [26]). Let $f, g : A \longrightarrow B$ be mappings. A pair (f, g) is said to commute proximally if for each $x, u, v \in A$,

$$d(u, fx) = d(v, gx) = d(A, B) \Rightarrow fv = gu. \tag{9}$$

Lemma 26 (see [19]). *Let (X, d) be a b-metric space with parameter $s \geq 1$. Assume that x_n and y_n are b-convergent to x and y, respectively. Then, we have the following:*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \longrightarrow \infty} d(x_n, y_n) \leq \limsup_{n \longrightarrow \infty} d(x_n, y_n) \leq s^2 d(x, y). \tag{10}$$

In particular, if $x = y$, then we have $\lim_{n \longrightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have the following:

$$\frac{1}{s} d(x, z) \leq \liminf_{n \longrightarrow \infty} d(x_n, z) \leq \limsup_{n \longrightarrow \infty} d(x_n, z) \leq s d(x, z). \tag{11}$$

Definition 27 (see [22]). A function $f : X \longrightarrow [0,\infty)$, where (X, d) is a b-metric space, is called lower semicontinuous if for all $x \in X$, and a sequence $\{x_n\}$ is b-convergent to x, and we have

$$f(x) \leq \liminf_{n \longrightarrow \infty} f(x_n). \tag{12}$$

Consider the following:

$\Psi = \{\psi : [0,\infty) \longrightarrow [0,\infty)$ such that ψ is continuous and nondecreasing function$\}$.

Also, we denote $\Phi = \{\phi : [0,\infty) \longrightarrow [0,\infty)$ such that ϕ is nondecreasing and lower semicontinuous, and

$$\phi(t) = 0 \Leftrightarrow t = 0\}. \tag{13}$$

Hao and Guan [22] proved the following common fixed point results for generalized weakly contractive mapping in b-metric spaces:

Theorem 28 (see [22]). *Let (X, d) be a complete b-metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ be given self-mappings satisfying g as injective and $f(X) \subset g(X)$ where $g(X)$ is closed. Suppose $\varphi : X \longrightarrow [0,\infty)$ is a lower semicontinuous function and $p \geq 2$ is a constant. If there are functions $\psi \in \Psi, \phi \in \Phi$ such that*

$$\begin{aligned} &\psi(s^p[d(fx, fy) + \varphi(fx) + \varphi(fy)]) \\ &\quad \leq \psi(m(x, y, d, f, g, \varphi)) - \phi(l(x, y, d, f, g, \varphi)), \end{aligned} \tag{14}$$

where

$$\begin{aligned} m(x, y, d, f, g, \varphi) = \max \Big\{ & d(gx, gy) + \varphi(gx) + \varphi(gy), \frac{1}{2}\{d(fx, gx) \\ & + \varphi(fx) + \varphi(gx) + d(fy, gy) + \varphi(fy) + \varphi(gy)\}, \\ & \cdot \frac{1}{2s}\{d(fx, gy) + \varphi(fx) + \varphi(gy) + d(fy, gx) \\ & + \varphi(fy) + \varphi(gx)\}\}, \end{aligned} \tag{15}$$

$$\begin{aligned} l(x, y, d, f, g, \varphi) = \max \{ & d(gx, gy) + \varphi(gx) + \varphi(gy), d(fy, gy) \\ & + \varphi(fy) + \varphi(gy)\}, \end{aligned} \tag{16}$$

then f and g have a unique coincidence point in X.

Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

3. Result and Discussion

Definition 29. Let (X, d) be a b-metric space and A and B be two nonempty subset of a b-metric space (X, d) with parameter $s \geq 1$ and $p \geq 2$ is a constant. A pair of map $f, g : A \longrightarrow B$ is said to be a generalized proximal weakly contractive mapping, if for all $x, y, h, t, r, m \in A$,

$$d(h, fx) = d(A, B), \tag{17}$$

$$d(t, fy) = d(A, B), \tag{18}$$

$$d(r, gx) = d(A, B), \tag{19}$$

$$d(m, gy) = d(A, B), \tag{20}$$

then

$$\begin{aligned}&\psi(s^p[d(h,t) + \varphi(h) + \varphi(t)])\\ &\quad \leq \psi(m_d(x, y, h, t, r, m, d, \varphi)) - \phi(l_d(x, y, h, t, r, m, d, \varphi)),\end{aligned} \tag{21}$$

where

$$\begin{aligned}&m_d(x, y, h, t, r, m, d, \varphi)\\ &\quad = \max\left\{d(r, m) + \varphi(r) + \varphi(m), \frac{1}{2}[d(h, r) + \varphi(h) + \varphi(r)\right.\\ &\qquad + d(t, m) + \varphi(t) + \varphi(m)], \frac{1}{2s}[d(h, m) + \varphi(h) + \varphi(m)\\ &\qquad \left.+ d(t, r) + \varphi(t) + \varphi(r)]\right\}.\end{aligned} \tag{22}$$

$l_d(x, y, h, t, r, m, d, \varphi) = \max\{d(r, m) + \varphi(r) + \varphi(m), d(t, m) + \varphi(t) + \varphi(m)\}$,$\psi \in \Psi$, $\phi \in \Phi$, and $\varphi : X \longrightarrow [0,\infty)$ is a lower semicontinuous function.

Theorem 30. *Let (A, B) be a pair of nonempty subsets of a complete b-metric space (X, d), and assume that A_0 and B_0 are nonempty such that A_0 is closed. Define a pair of mapping $f, g : A \longrightarrow B$ satisfying the following conditions:*

(i) f and g are generalized proximal weakly contractive mapping

(ii) $f(A_0) \subseteq B_0$, and $f(A_0) \subset g(A_0)$

(iii) f and g are continuous mapping

(iv) f and g are commute proximity

Then, f and g have a unique common best proximity point.

Proof. We prove the existence of common best proximity point.

Let $x_0 \in A_0$. Since $f(A_0) \subset g(A_0)$, there exists $x_1 \in A_0$ such that

$$fx_0 = gx_1. \tag{23}$$

Also, $x_1 \in A_0$. Since $f(A_0) \subset g(A_0)$, there exists $x_2 \in A_0$ such that

$$fx_1 = gx_2. \tag{24}$$

Continuing this process in a similar fashion, obtain the sequence $\{x_n\}$ and $\{x_{n+1}\}$ in A_0 such that

$$fx_n = gx_{n+1}, \tag{25}$$

for each $n \geq 0$.

Since $f(A_0) \subseteq B_0$ and A_0 is nonempty set, there exists $u_n \in A_0$ such that

$$d(u_n, fx_n) = d(A, B), \tag{26}$$

for all $n \geq 0$.

Further, we obtain that

$$d(A, B) = d(u_n, fx_n) = d(u_n, gx_{n+1}), \tag{27}$$

for all $n \geq 0$.

Our first goal is to show that $fu = gu$, for some $u \in A_0$.

Suppose that $u_n = u_{n+1}$, for some $n \geq 0$, by (2) and (3), we get that

$$d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u_n, fx_n) = d(u_n, gx_{n+1}). \tag{28}$$

Since f and g commute proximally, $fu_n = u_{n+1} = gu_n$, and so we are done.

Assume that $u_n \neq u_{n+1}$, for all $n \geq 0$. From (3), note that

$$d(u_n, fx_n) = d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u_{n-1}, gx_n) = d(u_n, gx_{n+1}), \tag{29}$$

for all $n \geq 1$. Since a pair (f, g) is generalized proximal weakly contractive map with $x = x_n$, $y = x_{n+1}$, we have that

$$\begin{aligned}&\psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}))\\ &\quad \leq \psi(s^p[d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})])\\ &\quad \leq \psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\ &\qquad - \phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)),\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&\psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad = \max\Big\{d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n), \frac{1}{2}\{d(u_n, u_{n-1}) + \varphi(u_n)\\
&\qquad + \varphi(u_{n-1}) + d(u_{n+1}, u_n) + \varphi(u_{n+1}) + \varphi(u_n)\}, \frac{1}{2s}\{d(u_n, u_n)\\
&\qquad + \varphi(u_n) + \varphi(u_n) + d(u_{n+1}, u_{n-1}) + \varphi(u_{n+1}) + \varphi(u_{n-1})\}\Big\}\\
&\quad \le \max\{d(u_n, u_{n-1}) + \varphi(u_n) + \varphi(u_{n-1}), d(u_n, u_{n+1}) + \varphi(u_n)\\
&\qquad + \varphi(u_{n+1})\},
\end{aligned}$$

$$\begin{aligned}
&\phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad = \max\{d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n), d(u_n, u_{n+1}) + \varphi(u_n)\\
&\qquad + \varphi(u_{n+1})\}.
\end{aligned} \tag{31}$$

If $d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}) > d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n)$, for some $n \in \mathbb{N}$, in view of (5)–(8), we have

$$\begin{aligned}
&\psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}))\\
&\quad \le \psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad - \phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad \le \psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}))\\
&\quad - \phi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})),
\end{aligned} \tag{32}$$

which implies $\phi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})) = 0$.
Hence, $u_n = u_{n+1}$, a contradiction.
Thus, we have

$$\begin{aligned}
&d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}) \le d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n),\\
&m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi) \le d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n),\\
&l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi) = d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n).
\end{aligned} \tag{33}$$

It follows from (10) that $\{d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})\}$ is a nonincreasing sequence, and so there exists $r \ge 0$ such that

$$\lim_{n\longrightarrow\infty}(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})) = r. \tag{34}$$

By (5), (11), and (12), we can obtain

$$\begin{aligned}
&\psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}))\\
&\quad \le \psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\qquad - \phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad \le \psi(d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n))\\
&\quad - \phi(d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n)).
\end{aligned} \tag{35}$$

Now assume that $r > 0$. Taking the upper limit as $n \longrightarrow \infty$ in (15), we have

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty}\psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}))\\
&\quad \le \limsup_{n\longrightarrow\infty}\psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad - \limsup_{n\longrightarrow\infty}\phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi))\\
&\quad \le \limsup_{n\longrightarrow\infty}\psi(d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n))\\
&\quad - \liminf_{n\longrightarrow\infty}\phi(d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n)),
\end{aligned} \tag{36}$$

which implies that $\psi(r) \le \psi(r) - \phi(r)$, a contradiction. Thus, we have

$$\lim_{n\longrightarrow\infty}(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})) = r = 0. \tag{37}$$

It follows that

$$\lim_{n\longrightarrow\infty} d(u_n, u_{n+1}) = 0, \lim_{n\longrightarrow\infty}\varphi(u_n) = 0. \tag{38}$$

Now, we claim that $\{u_n\}$ is a Cauchy sequence.

Suppose contradiction, that is, $\{u_n\}$, is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ such that there are subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ so that for all $k \in \mathbb{N}$ with $n_k > m_k > k$, we obtain

$$\begin{aligned}
&\varepsilon \le d(u_{m_k}, u_{n_k}),\\
&d(u_{m_k}, u_{n_k-1}) < \varepsilon.
\end{aligned} \tag{39}$$

By triangular inequality in b-metric space and (19) and (20), we have

$$\varepsilon \le d(u_{m_k}, u_{n_k}) \le sd(u_{m_k}, u_{n_k-1}) + sd(u_{n_k-1}, u_{n_k}) < s\varepsilon + sd(u_{n_k-1}, u_{n_k}). \tag{40}$$

Taking the upper limit as $k \longrightarrow \infty$ in the above inequality, we have

$$\begin{aligned}
&\varepsilon \le \limsup_{n\longrightarrow\infty} d(u_{m_k}, u_{n_k}) \le s\varepsilon,\\
&\frac{\varepsilon}{s} \le \limsup_{n\longrightarrow\infty} d(u_{m_k}, u_{n_{k-1}}) \le \varepsilon.
\end{aligned} \tag{41}$$

Also, we have

$$\begin{aligned}
\varepsilon &\le d(u_{m_k}, u_{n_k}) \le sd(u_{m_k}, u_{m_k-1}) + sd(u_{m_k-1}, u_{n_k})\\
&\le sd(u_{m_k}, u_{m_k-1}) + s^2 d(u_{m_k-1}, u_{m_k}) + s^2 d(u_{m_k}, u_{n_k})\\
&\le sd(u_{m_k}, u_{m_k-1}) + s^2 d(u_{m_k-1}, u_{m_k}) + s^3\varepsilon.
\end{aligned} \tag{42}$$

Then, by taking the upper limit as $k \longrightarrow \infty$ in (42), we have

$$\varepsilon \le \limsup_{n\longrightarrow\infty} sd(u_{m_k-1}, u_{n_k}) \le s^3\varepsilon, \tag{43}$$

which implies

$$\frac{\varepsilon}{s} \le \limsup_{n\longrightarrow\infty} d\left(u_{m_k-1}, u_{n_k}\right) \le s^2\varepsilon. \tag{44}$$

It is from

$$\begin{aligned}\varepsilon \le d\left(u_{m_k}, u_{n_k}\right) &\le sd\left(u_{m_k}, u_{m_k-1}\right) + sd\left(u_{m_k-1}, u_{n_k}\right)\\ &\le sd\left(u_{m_k}, u_{m_k-1}\right) + s^2 d\left(u_{m_k-1}, u_{n_k-1}\right) + s^2 d\left(u_{n_k-1}, u_{n_k}\right)\\ &\le sd\left(u_{m_k}, u_{m_k-1}\right) + s^3 d\left(u_{m_k-1}, u_{m_k}\right) + s^3 d\left(u_{m_k}, u_{n_k-1}\right)\\ &\le sd\left(u_{m_k}, u_{m_k-1}\right) + s^3 d\left(u_{m_k-1}, u_{m_k}\right) + s^3\varepsilon.\end{aligned} \tag{45}$$

By taking the upper limit as $k \longrightarrow \infty$ in (43), we have

$$\frac{\varepsilon}{s^2} \le \limsup_{n\longrightarrow\infty} d\left(u_{m_k-1}, u_{n_k-1}\right) \le s\varepsilon. \tag{46}$$

In similar fashion by taking the lower limit, we can obtain

$$\begin{aligned}\varepsilon &\le \liminf_{n\longrightarrow\infty} d\left(u_{m_k}, u_{n_k}\right) \le s\varepsilon,\\ \frac{\varepsilon}{s} &\le \liminf_{n\longrightarrow\infty} d\left(u_{m_k}, u_{n_{k-1}}\right) \le \varepsilon,\\ \frac{\varepsilon}{s} &\le \liminf_{n\longrightarrow\infty} d\left(u_{m_k-1}, u_{n_k}\right) \le s^2\varepsilon,\\ \frac{\varepsilon}{s^2} &\le \liminf_{n\longrightarrow\infty} d\left(u_{m_k-1}, u_{n_k-1}\right) \le s\varepsilon.\end{aligned} \tag{47}$$

Since $\{u_{m_k}\}$ and $\{u_{n_k}\}$ satisfy equations (26) and (27), we obtain that

$$\begin{aligned}d\left(u_{n_k}, fx_{n_k}\right) &= d(A,B) = d\left(u_{n_{k-1}}, gx_{n_k}\right),\\ d\left(u_{m_k}, fx_{m_k}\right) &= d(A,B) = d\left(u_{m_{k-1}}, gx_{m_k}\right),\end{aligned} \tag{48}$$

for each $k \in \mathbb{N}$. Since f and g are generalized proximal weakly contractive mapping with $x = x_{n_k}$ and $y = x_{m_k}$, we have

$$\begin{aligned}&\psi\left(d\left(u_{n_k}, u_{m_k}\right) + \varphi\left(u_{n_k}\right) + \varphi\left(u_{m_k}\right)\right)\\ &\quad\le \psi\left(m_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\right)\\ &\qquad- \phi\left(l_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\right).\end{aligned} \tag{49}$$

From the definition, we have

$$\begin{aligned}&m_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\\ &\le \max\left\{d\left(u_{m_{k-1}}, u_{n_k-1}\right) + \varphi\left(u_{m_k-1}\right) + \varphi\left(u_{n_k-1}\right), \frac{1}{2}\left\{d\left(u_{n_k}, u_{n_{k-1}}\right)\right.\right.\\ &\quad+ \varphi\left(u_{n_k}\right) + \varphi\left(u_{n_{k-1}}\right) + d\left(u_{m_k}, u_{m_{k-1}}\right) + \varphi\left(u_{m_k}\right) + \varphi\left(u_{m_{k-1}}\right)\Big\}, \frac{1}{2s}\left\{d\left(u_{n_k}, u_{m_{k-1}}\right)\right.\\ &\quad\left.\left.+ \varphi\left(u_{n_k}\right) + \varphi\left(u_{m_{k-1}}\right) + d\left(u_{n_{k-1}}, u_{m_k}\right) + \varphi\left(u_{n_{k-1}}\right) + \varphi\left(u_{m_{k-1}}\right)\right\}\right\}.\end{aligned} \tag{50}$$

Taking the upper limit as $k \longrightarrow \infty$, we obtain

$$\begin{aligned}&\limsup_{n\longrightarrow\infty} m_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\\ &\quad\le \max\left\{s\varepsilon, 0, \frac{\varepsilon + s^2\varepsilon}{2s}\right\} = s\varepsilon.\end{aligned} \tag{51}$$

Also, we have

$$\begin{aligned}&l_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\\ &\quad= \max\left\{d\left(u_{m_{k-1}}, u_{n_k-1}\right) + \varphi\left(u_{m_k-1}\right) + \varphi\left(u_{n_k-1}\right), d\left(u_{m_k}, u_{m_k-1}\right)\right.\\ &\qquad\left.+ \varphi\left(u_{m_k}\right) + \varphi\left(u_{m_k-1}\right)\right\}.\end{aligned} \tag{52}$$

By taking the lower limit as $k \longrightarrow \infty$, we have

$$s\varepsilon \ge \liminf_{n\longrightarrow\infty} l_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right) \ge \frac{\varepsilon}{s^2}. \tag{53}$$

By applying generalized proximal weakly contractive mapping with $x = x_{n_k}$ and $y = x_{m_k}$, we have

$$\begin{aligned}\psi(s\varepsilon) \le \psi(s^p\varepsilon) &\le \psi\left(s^p \limsup_{n\longrightarrow\infty}\left[d\left(u_{m_k}, u_{n_k}\right) + \varphi\left(u_{m_k}\right) + \varphi\left(u_{n_k}\right)\right]\right)\\ &\le \psi\left(\limsup_{n\longrightarrow\infty} m_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\right)\\ &\quad- \liminf_{n\longrightarrow\infty}\phi\left(l_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\right)\\ &\le \psi(s\varepsilon) - \phi\left(\liminf_{n\longrightarrow\infty} l_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right)\right),\end{aligned} \tag{54}$$

which implies that

$$\liminf_{n\longrightarrow\infty} l_d\left(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi\right) = 0, \tag{55}$$

a contradiction to (53). Hence, the sequence $\{u_n\}$ is Cauchy. Since A_0 be a closed subset of the complete b-metric space X, there exists $u \in A_0$ such that

$$\lim_{n\longrightarrow\infty} u_n = u. \tag{56}$$

By the definition of φ, we have

$$\varphi(u) \le \liminf_{n\longrightarrow\infty}\varphi(u_n) = 0 \Rightarrow \varphi(u) = 0. \tag{57}$$

Consider, by (2) and (3), that

$$d(u_n, fx_n) = d(u_{n-1}, gx_n) = d(A,B). \tag{58}$$

Since f and g are commute proximally,

$$fu_{n-1} = gu_n, \tag{59}$$

for all $n \in \mathbb{N}$. By continuity of f and g,

$$fu = \lim_{n\longrightarrow\infty} fu_{n-1} = \lim_{n\longrightarrow\infty} gu_n = gu. \tag{60}$$

Now, we claim the existence of common best proximity point of f and g. Since $f(A_0) \subseteq B_0$, there exists $x^\star \in A_0$ such that

$$d(x^\star, fu) = d(x^\star, gu) = d(A, B). \tag{61}$$

By the assumption that f and g commute proximally, $fx^\star = gx^\star$.

According to the assumption that $f(A_0) \subseteq B_0$, there exists $z^\star \in A_0$ such that

$$d(z^\star, fx^\star) = d(z^\star, gx^\star) = d(A, B). \tag{62}$$

Next, we claim that $x^\star = z^\star$. Suppose that $x^\star \neq z^\star$, that is, $d(x^\star, z^\star)) > 0$. By applying generalized proximal weakly contractive mapping with $x = u$ and $y = x^\star$, we observe that

$$\begin{aligned}
&\psi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)) \\
&\quad \le \psi(s^p[d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)]) \\
&\quad \le \psi(m_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi)) \\
&\qquad - \phi(l_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi)),
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
&m_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi) \\
&\quad = \max\Bigg\{d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star), \frac{1}{2}\{d(x^\star, x^\star) + \varphi(x^\star) \\
&\qquad + \varphi(x^\star) + d(z^\star, z^\star) + \varphi(z^\star) + \varphi(z^\star)\}, \frac{1}{2s}\{d(x^\star, z^\star) \\
&\qquad + \varphi(x^\star) + \varphi(z^\star) + d(z^\star, x^\star) + \varphi(z^\star) + \varphi(x^\star)\}\Bigg\} \\
&\quad \le d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star),
\end{aligned} \tag{64}$$

$$\begin{aligned}
&l_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi) \\
&\quad = \max\{d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star), d(z^\star, z^\star) + \varphi(z^\star) + \varphi(z^\star)\} \\
&\quad = d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star).
\end{aligned} \tag{65}$$

From (63)–(65), we have

$$\begin{aligned}
&\psi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)) \\
&\quad \le \psi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)) - \phi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)),
\end{aligned} \tag{66}$$

which implies

$$\begin{aligned}
&d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star) = 0, \\
&\Rightarrow d(x^\star, z^\star) = 0, \\
&\varphi(x^\star) = 0.
\end{aligned} \tag{67}$$

This contradicts the assumption $x^\star \neq z^\star$. Thus, $x^\star = z^\star$. Hence,

$$d(x^\star, fx^\star) = d(A, B) = d(x^\star, gx^\star). \tag{68}$$

That is, the element $x^\star \in A$ is a common best proximity point of f and g.

Finally, we have to show that the point $x^\star$ is unique.

Let $y^\star \in A$ be another common best proximity point of f and g. Then,

$$d(x^\star, fx^\star) = d(y^\star, fy^\star) = d(A, B) = d(x^\star, gx^\star) = d(y^\star, gy^\star). \tag{69}$$

Since f and g are generalized proximal weakly contractive mapping, we obtain that

$$\begin{aligned}
&\psi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)) \\
&\quad \le \psi(s^p[d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)]) \\
&\quad \le \psi(m_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi)) \\
&\quad - \phi(l_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi)),
\end{aligned} \tag{70}$$

where

$$\begin{aligned}
&m_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi) \\
&\quad = \max\Bigg\{d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star), \frac{1}{2}\{d(x^\star, x^\star) + \varphi(x^\star) \\
&\qquad + \varphi(x^\star) + d(y^\star, y^\star) + \varphi(y^\star) + \varphi(y^\star)\}, \frac{1}{2s}\{d(x^\star, y^\star) \\
&\qquad + \varphi(x^\star) + \varphi(y^\star) + d(y^\star, x^\star) + \varphi(y^\star) + \varphi(x^\star)\}\Bigg\} \\
&\quad \le d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star),
\end{aligned} \tag{71}$$

$$\begin{aligned}
&l_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi) \\
&\quad = \max\{d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star), d(y^\star, y^\star) + \varphi(y^\star) + \varphi(y^\star)\} \\
&\quad = d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star).
\end{aligned} \tag{72}$$

Now, from (71) and (72), we have

$$\begin{aligned}
&\psi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)) \\
&\quad \le \psi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)) - \phi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)).
\end{aligned} \tag{73}$$

By the properties of ϕ and from (73), we have

$$\begin{aligned}
&d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star) = 0, \\
&\Rightarrow d(x^\star, y^\star) = 0, \varphi(x^\star) = 0,
\end{aligned} \tag{74}$$

which contradict the supposition that $x^\star \neq y^\star$. Thus, $x^\star = y^\star$.

Therefore, f and g have a unique common best proximity point.

The proof is completed. □

Example 31. Let $X = \mathbb{R}^2$ and $d : X \times X \longrightarrow [0,\infty)$ be defined by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|^2 + |x_2 - y_2|^2$, for all $(x_1, x_2), (y_1, y_2) \in X$ and (X, d) is a complete b-metric space with parameter $s = 2$.

Suppose

$$A = \{(x, 0): 0 \le x \le 1\},$$
$$B = \{(x, 1): 0 \le x \le 1\}. \tag{75}$$

Let $f, g : A \longrightarrow B$ be the mapping defined by

$$f(x, 0) = \left(\frac{x}{8}, 1\right),$$
$$g(x, 0) = \left(7\left(\frac{x}{8}\right), 1\right). \tag{76}$$

$\varphi : X \longrightarrow [0,\infty)$, defined by $\varphi(x, 0) = x^2$, and define a mapping $\psi, \phi : [0,\infty) \longrightarrow [0,\infty)$ with $\psi(t) = t$, and $\phi(t) = 35t/98$. Clearly, φ is lower semicontinuous function, and ψ is continuous and nondecreasing function. Further, ϕ is nondecreasing and lower semicontinuous, and $\phi(t) = 0 \Leftrightarrow t = 0$.

$d(A, B) = \inf\{d((x, 0), (x, 1)): (x, 0) \in A, (x, 1) \in B\}$, which implies that $d(A, B) = \inf\{|x - x|^2 + |0 - 1|^2\} = 1$ and implies that $d(A, B) = 1$.

Notice that f and g are continuous. Now, we check that f and g are generalized proximal weakly contractive mapping.

In fact, for all $(x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0) \in A$, we have

$d((h, 0), f(x, 0)) = d(A, B)$ implies that $d((h, 0), (x/8, 1)) = 1$, which implies that $h = x/8$

$d((t, 0), f(y, 0)) = d(A, B)$; this implies $d((t, 0), (y/8, 1)) = 1$, which implies that $t = y/8$

$d((r, 0), g(x, 0)) = d(A, B)$ implies that $d((r, 0), (7(x/8), 1)) = 1$, implies that $r = 7(x/8)$

$d((m, 0), g(y, 0)) = d(A, B)$ implies that $d((m, 0), (7(y/8), 1)) = 1$, implies that $m = 7(y/8)$

Then, by applying the generalized proximal weakly contractive mapping definition with $s = p = 2$, we have the following:

$$\begin{aligned}&\psi(s^p[d((h, 0), (t, 0)) + \varphi(h, 0) + \varphi(t, 0)])\\ &\quad \le \psi(m_d((x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0), d, \varphi))\\ &\qquad - \phi(l_d((x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0), d, \varphi)).\end{aligned} \tag{77}$$

Now, we have

$$\begin{aligned}&\psi(s^p[d((h, 0), (t, 0)) + \varphi(h, 0) + \varphi(t, 0)])\\ &= 2^2.\left[d\left(\left(\frac{x}{8}, 0\right), \left(\frac{y}{8}, 0\right)\right) + \varphi\left(\frac{x}{8}, 0\right) + \varphi\left(\frac{y}{8}, 0\right)\right)\right]\\ &= 4.\left[d\left(\left(\frac{x}{8}, 0\right), \left(\frac{y}{8}, 0\right)\right) + \left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2\right]\\ &= 4.\left[\left|\frac{x}{8} - \frac{y}{8}\right|^2 + |0 - 0|^2 + \left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2\right.\\ &= 4.\left[\left(\frac{x}{8} - \frac{y}{8}\right)^2 + \left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2\right] \le 4.\frac{1}{64}.2(x^2 + y^2) = \frac{1}{8}(x^2 + y^2),\end{aligned}$$

$$\begin{aligned}&\psi(m_d((x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0), d, \varphi))\\ &= \max\left\{d((r, 0), (m, 0)) + \varphi(r, 0) + \varphi(m, 0), \frac{1}{2}\{d((h, 0), (r, 0)) + \varphi(h, 0)\right.\\ &\quad + d((t, 0), (m, 0)) + \varphi(t, 0) + \varphi(m, 0)\}, \frac{1}{2s}\{d((h, 0), (m, 0)) + \varphi(h, 0)\\ &\quad \left.+ \varphi(m, 0) + d((t, 0), (r, 0)) + \varphi(t, 0) + \varphi(r, 0)\}\right\}\\ &\ge \psi\left(\frac{1}{2}[d((h, 0), (r, 0)) + \varphi(h, 0) + \varphi(r, 0) + d((t, 0), (m, 0)) + \varphi(t, 0) + \varphi(m, 0)]\right)\\ &= \frac{1}{2}\left[d\left(\left(\frac{x}{8}, 0\right), \left(\frac{7x}{8}, 0\right) + \varphi\left(\frac{x}{8}, 0\right) + \varphi\left(\frac{7x}{8}, 0\right) + d\left(\left(\frac{y}{8}, 0\right), \left(\frac{7y}{8}, 0\right)\right)\right.\right.\\ &\quad \left.\left.+ \varphi\left(\frac{y}{8}, 0\right) + \varphi\left(\frac{7y}{8}, 0\right)\right]\right.\\ &= \frac{1}{2}\left[\left|\frac{x}{8} - \frac{7x}{8}\right|^2 + |0 - 0|^2 + \left(\frac{x}{8}\right)^2 + \left(\frac{7x}{8}\right)^2 + \left|\frac{y}{8} - \frac{7y}{8}\right|^2 + |0 - 0|^2 + \left(\frac{y}{8}\right)^2 + \left(\frac{7y}{8}\right)^2\right]\\ &= \frac{1}{2}.\frac{1}{64}\left(36x^2 + x^2 + 49x^2 + 36y^2 + y^2 + 49y^2\right) = \frac{1}{2}.\frac{1}{64}\left(86x^2 + 86y^2\right)\\ &= \frac{43}{64}\left(x^2 + y^2\right),\end{aligned}$$

$$\begin{aligned}&\phi(l_d((x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0), d, \varphi))\\ &= \phi(\max\{d((r, 0), (m, 0)) + \varphi(r, 0) + \varphi(m, 0), d((t, 0), (m, 0)) + \varphi(t, 0) + \varphi(m, 0)\})\\ &= \frac{35}{98}\max\left\{d\left(\left(\frac{7x}{8}, 0\right), \left(\frac{7y}{8}, 0\right)\right) + \varphi\left(\frac{7x}{8}, 0\right)\right.\\ &\quad \left.+ \varphi\left(\frac{7y}{8}, 0\right), d\left(\left(\frac{y}{8}, 0\right), \left(\frac{7y}{8}, 0\right)\right) + \varphi\left(\frac{y}{8}, 0\right) + \varphi\left(\frac{7y}{8}, 0\right)\right\}\\ &= \frac{35}{98}\max\left\{\left|\frac{7x}{8} - \frac{7y}{8}\right|^2 + |0 - 0|^2 + \left(\frac{7x}{8}\right)^2 + \left(\frac{7y}{8}\right)^2, \left|\frac{y}{8} - \frac{7y}{8}\right|^2\right.\\ &\quad \left.+ |0 - 0|^2 + \left(\frac{y}{8}\right)^2 + \left(\frac{7y}{8}\right)^2\right\} = \frac{35}{98}\max\left\{\frac{49}{32}(x^2 + y^2), \frac{43}{32}y^2\right\} \le \frac{35}{64}(x^2 + y^2).\end{aligned} \tag{78}$$

According to above inequalities, we get that

$$\begin{aligned}&\psi(s^p[d((h, 0), (t, 0)) + \varphi(h, 0) + \varphi(t, 0)])\\ &\quad \le \frac{1}{8}(x^2 + y^2) = \frac{43}{64}(x^2 + y^2) - \frac{35}{64}(x^2 + y^2)\\ &\quad \le \psi(m_d((x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0), d, \varphi))\\ &\qquad - \phi(l_d((x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0), d, \varphi)).\end{aligned} \tag{79}$$

Hence, f and g are generalized proximal weakly contractive mappings.

Next, consider, by the definition of A_0, B_0, that $A_0 = A$ and $B_0 = B$; thus, $f(A_0)$ and $g(A_0) \subseteq B_0$. Additionally, $f(A_0) = \{(x, 1): 0 \le x \le 1/8\} \subset \{(x, 1): 0 \le x \le 7/8\} = g(A_0)$.

Now, it remains to show that f and g commute proximally. Let $x, u, v \in A$ such that

$$d(u, fx) = d(v, gx) = d(A, B). \tag{80}$$

Consequently, $x = (\hat{x}, 0)$, $u = (\hat{u}, 0)$, and $v = (\hat{v}, 0)$, where

$\widehat{u} = x/8$ and $\widehat{v} = 7\widehat{x}/8$. Thus,

$$\begin{aligned} fx &= f(\widehat{x}, 0) = \left(\frac{\widehat{x}}{8}, 1\right), \\ gx &= g(\widehat{x}, 0) = \left(\frac{7\widehat{x}}{8}, 1\right), \\ d(u, fx) &= d((\widehat{u}, 0), f(\widehat{x}, 0)) = d\left(\left(\frac{\widehat{x}}{8}, 0\right), \left(\frac{\widehat{x}}{8}, 1\right)\right) = 1 = d(A, B), \\ d(v, gx) &= d((\widehat{v}, 0), g(\widehat{x}, 0)) = d\left(\left(\frac{7\widehat{x}}{8}, 0\right), \left(\frac{7\widehat{x}}{8}, 1\right)\right) = 1 = d(A, B). \end{aligned} \tag{81}$$

Therefore,

$$d(u, fx) = d(v, gx) = d(A, B). \tag{82}$$

Now, we claim that

$$\begin{aligned} &fv = gu, \\ &gu = g(\widehat{u}, 0) = g\left(\frac{\widehat{x}}{8}, 0\right) = \left(\frac{7\widehat{x}}{64}, 1\right), \\ &fv = f(\widehat{v}, 0) = f\left(\frac{7\widehat{x}}{8}, 0\right) = \left(\frac{7\widehat{x}}{64}, 1\right), \end{aligned} \tag{83}$$

which implies $fv = gu$.

Hence, $d(u, fx) = d(v, gx) = d(A, B) \Rightarrow fv = gu$.

Therefore, f and g are commute proximally.

Finally, by Theorem 30, we can conclude that there is a unique common best proximity point of the pair (f, g). In fact, the point $(0, 0)$ is the unique common best proximity point of (f, g).

To show this, there exists $(x^*, 0) \in A$ such that

$$d((x^*, 0), f(x^*, 0)) = d((x^*, 0), g(x^*, 0)) = d(A, B) = 1, \tag{84}$$

where $(x^*, 0)$ is common best proximity point of f and g. Now find x^*

$$d((x^*, 0), f(x^*, 0)) = d(A, B) = 1, \tag{85}$$

and this implies that

$$d\left((x^*, 0), \left(\frac{x^*}{8}, 1\right)\right) = 1, \tag{86}$$

implying that

$$\left|x^* - \frac{x^*}{8}\right|^2 + |0 - 1|^2 = 1. \tag{87}$$

From this, we get

$$\left|x^* - \frac{x^*}{8}\right|^2 = 0. \tag{88}$$

Hence, $x^* = 0$, and also, from

$$d((x^*, 0), g(x^*, 0)) = d(A, B) = 1, \tag{89}$$

we have

$$d\left((x^*, 0), \left(\frac{7x^*}{8}, 1\right)\right) = 1, \tag{90}$$

implying that

$$\left|x^* - \frac{7x^*}{8}\right|^2 + |0 - 1|^2 = 1. \tag{91}$$

From this, we get

$$\left|x^* - \frac{7x^*}{8}\right|^2 = 0. \tag{92}$$

Hence, $x^* = 0$.

Therefore, the point $(x^*, 0) = (0, 0) \in A$ is a unique common best proximity point of f and g.

If $\varphi = 0$ in Theorem 30, we can get the following result:

Corollary 32. *Let (A, B) be a pair of nonempty subsets of a complete b-metric space (X, d), and assume that A_0 and B_0 are nonempty such that A_0 is closed. Define a pair of mapping $f, g : A \longrightarrow B$ satisfying the following conditions:*

(i) For all $x, y, h, t, r, m \in A$,

$$\begin{aligned} d(h, fx) &= d(A, B), \\ d(t, fy) &= d(A, B), \\ d(r, gx) &= d(A, B), \\ d(m, gy) &= d(A, B), \end{aligned} \tag{93}$$

then

$$\begin{aligned} \psi(s^p d(h, t)) \leq \psi(&m_d(x, y, h, t, r, m, d)) \\ &- \phi(l_d(x, y, h, t, r, m, d)), \end{aligned} \tag{94}$$

where

$$\begin{aligned} m_d(x, y, h, t, r, m, d) &= \max\left\{d(r, m), \frac{1}{2}[d(h, r) + d(t, m)], \frac{1}{2s}[d(h, m) + d(t, r)]\right\}, \\ l_d(x, y, h, t, r, m, d) &= \max\{d(r, m), d(t, m)\}, \\ &\psi \in \Psi, \phi \in \Phi, \end{aligned} \tag{95}$$

(ii) $f(A_0) \subseteq B_0$ and $f(A_0) \subset g(A_0)$.

(iii) f and g are continuous mapping

(iv) f and g are commute proximity

Then, f and g have a unique common best proximity point.

If we consider the corresponding problem in the setting of metric space, that is, $s = 1$ in Theorem 30, we can obtain the following:

Corollary 33. *Let (A, B) be a pair of nonempty subsets of a complete b-metric space (X, d) and assume that A_0 and B_0 are nonempty such that A_0 is closed. Define a pair of mapping $f, g : A \longrightarrow B$ satisfying the following conditions:*

(i) For all $x, y, h, t, r, m \in A$,

$$\begin{aligned} d(h, fx) &= d(A, B), \\ d(t, fy) &= d(A, B), \\ d(r, gx) &= d(A, B), \\ d(m, gy) &= d(A, B), \end{aligned} \tag{96}$$

then

$$\begin{aligned} &\psi(d(h, t) + \varphi(h) + \varphi(t)) \\ &\quad \leq \psi(m_d(x, y, h, t, r, m, d, \varphi)) - \phi(l_d(x, y, h, t, r, m, d, \varphi)), \end{aligned} \tag{97}$$

where

$$\begin{aligned} m_d(x, y, h, t, r, m, d, \varphi) = \max \Big\{ & d(r, m) + \varphi(r) + \varphi(m), \frac{1}{2}[d(h, r) + \varphi(h) + \varphi(r) \\ & + d(t, m) + \varphi(t) + \varphi(m)], \frac{1}{2}[d(h, m) + \varphi(h) + \varphi(m) \\ & + d(t, r) + \varphi(t) + \varphi(r)]\}, \end{aligned}$$

$$l_d(x, y, h, t, r, m, d, \varphi) = \max \{d(r, m) + \varphi(r) + \varphi(m), d(t, m) + \varphi(t) + \varphi(m)\}, \tag{98}$$

is the same as Theorem 30, $\psi \in \Psi$, $\phi \in \Phi$, and $\varphi : X \longrightarrow [0, \infty)$ is a lower semicontinuous function.

$$f(A_0) \subseteq B_0, and f(A_0) \subset g(A_0) \tag{99}$$

(ii) f and g are continuous mapping

(iii) f and g are commute proximity

Then, f and g have a unique common best proximity point.

4. Conclusion

This study is concerned with the existence and uniqueness of common best proximity point for generalized proximal weakly contractive mapping in complete b-metric spaces, and in this study, we have defined the notion of generalized proximal weakly contractive mapping in b-metric spaces.

Data Availability

All datas are included.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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