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# A New Approach to Dual Jacobsthal Split Quaternions with Different Polar Representation

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***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

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## Abstract

In this paper, we introduce split quaternions with components including dual Jacobsthal and dual Jacobsthal-Lucas number sequences. By using Binet's formulas of these type split quaternions we give an explicit form of classic polar representations of them, after that we demonstrate a new polar representation by using Cayley-Dikson's notation of split quaternions which is based on two complex numbers. Some fundamental properties and identities for these type of split quaternions are studied. In further the current paper, it would be valuable to replicate similar approaches polar representationin with dual Jacobsthal Split quaternions.

*Keywords: Dual Jacobsthal split quaternions; Dual Jacobsthal-Lucas split quaternions; Polar representation.*

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# 1 Introduction

Quaternions have found extensive applications in the field of 3D computer graphics, robotics, and physics, particularly in the representation and computation of rotations and orientations [1]. They offer a compact and efficient way to represent 3D rotations without suffering from gimbal lock, an issue prevalent with Euler angles [2]. Moreover, quaternions are used in the analysis of spatial rotations in physics and in the control mechanisms of spacecraft and robotic arms due to their ability to interpolate rotations smoothly and efficiently [3]. Quaternions, extension of complex numbers was introduced by W. R. Hamilton as the following quadruple

$$Q = \gamma_0 + \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ and $i_1^2 = i_2^2 = i_3^2 = -1, i_1 i_2 = -i_2 i_1 = i_3$. Any quaternion can be write as

$$Q = (\gamma_0 + \gamma_1 i_1) + i_2(\gamma_2 - \gamma_3 i_1)$$

where $\gamma_0 + \gamma_1 i_1$ and $\gamma_2 - \gamma_3 i_1$ are complex numbers, therefore quaternions are one of the hyper complex numbers, [4–8].

J. Cokle defined the split quaternion as $\hat{Q} = \delta_0 + \delta_1 i_1 + \delta_2 i_2 + \delta_3 i_3$ with $\delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$ and quatenionic units $i_1, i_2, i_3$ satisfy

$$i_1^2 = -i_2^2 = -i_3^2 = -1,\ i_1 i_2 = -i_2 i_1 = i_3 \tag{1.1}$$

and can be shown as $\hat{Q} = S_{\hat{Q}} + V_{\hat{Q}}$, $\hat{Q}$ with $S_{\hat{Q}} = 0$ is a pure split quaternion, [9–13].
Let $\hat{Q}_1 = S_{\hat{Q}_1} + V_{\hat{Q}_1}$ and $\hat{Q}_2 = S_{\hat{Q}_2} + V_{\hat{Q}_2}$ be split quaternions, then addition and multiplication are

$$\hat{Q}_1 + \hat{Q}_2 = (S_{\hat{Q}_1} + S_{\hat{Q}_2}) + (V_{\hat{Q}_1} + V_{\hat{Q}_2})$$
$$\hat{Q}_1\hat{Q}_2 = S_{\hat{Q}_1} S_{\hat{Q}_2} + \langle V_{\hat{Q}_1}, V_{\hat{Q}_2}\rangle + S_{\hat{Q}_1} V_{\hat{Q}_1} + S_{\hat{Q}_2} V_{\hat{Q}_2} + V_{\hat{Q}_1} \times V_{\hat{Q}_2}$$

respectively, where $\langle\ ,\ \rangle$ and $\times$ are inner and cross products in Minkowsky space $\mathbb{E}_1^3$. The conjugate and norm of $\hat{Q} = \delta_0 + \delta_1 i_1 + \delta_2 i_2 + \delta_3 i_3$ are respectively as $\overline{\hat{Q}} = \delta_0 - \delta_1 i_1 - \delta_2 i_2 - \delta_3 i_3$ and

$$N(\hat{Q}) = \sqrt{|\hat{Q}\overline{\hat{Q}}|} = \sqrt{|\delta_0^2 + \delta_1^2 - \delta_2^2 - \delta_3^2|} \tag{1.2}$$

if $N(\hat{Q}) = 1$, then $\hat{Q}$ is a unit split quaternion, for any split quaternion $\hat{Q}$ with $N(\hat{Q}) \neq 0$, $\frac{\hat{Q}}{N(\hat{Q})}$ is a unit split quaternion. Let $I_{\hat{Q}} = \delta_0^2 + \delta_1^2 - \delta_2^2 - \delta_3^2$, then $\hat{Q}$ is space-like, time-like and light-like if $I_{\hat{Q}} > 0$, $I_{\hat{Q}} < 0$ and $I_{\hat{Q}} = 0$ respectively, the multiplicative inverse of $\hat{Q}$ is $\hat{Q}^{-1} = \frac{\overline{\hat{Q}}}{N(\hat{Q})^2}$ and there is no inverse for light-like split quaternion. The Cayley-Dickson's form of a split quaternion $\hat{Q}$ is $\hat{Q} = (\delta_0 + \delta_1 i_1) + (\delta_2 + \delta_3 i_1) i_2$ which is based on two complex numbers.

Dual numbers, extension of real numbers, first were defined by W. K. Clifford in 1873 as:

$$\mathbb{D} = \{A = a_1 + \varepsilon a_2 \mid a_1, a_2 \in \mathbb{R}, \varepsilon \neq 0, \varepsilon^2 = 0\}$$

This new number system is commutative and associative algebra over real numbers wich has dimension two. Kotelnikov initiated the study of the first applications of dual numbers. Eduard Study used these numbers and associated vectors in line geometry and kinematics, [14, 15]. According to the following operation

$$(a_1 + \varepsilon a_2) + (b_1 + \varepsilon b_2) = (a_1 + b_1) + \varepsilon(a_2 + b_2)$$

$$(a_1 + \varepsilon a_2)(b_1 + \varepsilon b_2) = a_1 b_1 + \varepsilon(a_1 b_2 + a_2 b_1)$$

algebra of dual numbers is a commutative ring, but not a field. The multiplicative inverse of $A$ is

$$A^{-1} = \frac{1}{a_1} - \varepsilon\frac{a_2}{a_1^2},\ a_1 \neq 0$$

and there is no inverse for pure dual numbers and hence this algebra of numbers is not field over real numbers. By using inverse of dual numbers we can define division operation of two dual numbers $A_1$ and $B_1$ as $A_1B_1^{-1}$ where $B_1$ is not a pure dual number and $B_1 \neq 0$. Dual angle between lines $d_1$ and $d_2$ in $\mathbb{R}^3$ is defined as $\Phi = \phi + \varepsilon\phi^*$, where $\phi$ is angle and $\phi^*$ is the shortest distance between these lines. If $f(x + \varepsilon x^*)$ is a dual function, then by Taylor expansion we have

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x) \tag{1.3}$$

where $f'(x)$ is the first derivative of $f(x)$, using Equality (1.3) we can write

$$\sqrt{a_1 + \varepsilon a_2} = \sqrt{a_1} + \varepsilon \frac{a_1}{2\sqrt{a_2}} \tag{1.4}$$

Dual split quaternion $\mathbb{Q}$ is defined as

$$\mathbb{Q} = A + Bi_1 + Ci_2 + Di_3$$

where $A, B, C, D$ are dual numbers and $i_1, i_2, i_3$ follows rules in (1.1), $\mathbb{Q}$ can be write as $\mathbb{Q} = \hat{Q} + \varepsilon\hat{Q}^*$, where $\hat{Q}, \hat{Q}^*$ are split quaternions and $\varepsilon^2 = 0$. Let $\mathbb{Q}_1 = \hat{Q}_1 + \varepsilon\hat{Q}_1^*$ and $\mathbb{Q}_2 = \hat{Q}_2 + \varepsilon\hat{Q}_2^*$ be dual split quaternions, then

$$\mathbb{Q}_1 + \mathbb{Q}_2 = (\hat{Q}_1 + \hat{Q}_2) + \varepsilon(\hat{Q}_1^* + \hat{Q}_2^*)$$
$$\mathbb{Q}_1\mathbb{Q}_2 = \hat{Q}_1\hat{Q}_2 + \varepsilon(\hat{Q}_1\hat{Q}_2^* + \hat{Q}_1^*\hat{Q}_2)$$

norm of $\mathbb{Q}$ is defined as $N(\mathbb{Q}) = \sqrt{A^2 + B^2 - C^2 - D^2}$, and dual split quaternion with $N(\mathbb{Q}) = 1$ is called unit dual split quaternion. Dual split quaternion $\mathbb{Q} = \hat{Q} + \varepsilon\hat{Q}^*$ is spacelike, timelike or lightlike if and only $\hat{Q}$ is spacelike, timelike and lightlike, respectively.
Let $\Phi = \phi + \varepsilon\phi^*$ be dual angle then the polar representation for spacelike $\mathbb{Q}$ is as

$$\mathbb{Q} = N(\mathbb{Q})(\sinh\Phi + \nu\cosh\Phi) \tag{1.5}$$

where $\nu = \frac{Bi_+Ci_2+Di_3}{\sqrt{-B^2+C^2+D^2}}$ is a unit pure dual split quaternion, $\sinh\Phi = \frac{D}{N(\mathbb{Q})}$ and $\cosh\Phi = \frac{\sqrt{|B^2-C^2-D^2|}}{N(\mathbb{Q})}$. The polar representation for timelike $\mathbb{Q}$ with spacelike vector part is

$$\mathbb{Q} = N(\mathbb{Q})(\cosh\Phi + \nu\sinh\Phi) \tag{1.6}$$

where $\nu = \frac{Bi_+Ci_2+Di_3}{\sqrt{-B^2+C^2+D^2}}$ is a unit pure dual split quaternion, $\cosh\Phi = \frac{D}{N(\mathbb{Q})}$ and $\sinh\Phi = \frac{\sqrt{|B^2-C^2-D^2|}}{N(\mathbb{Q})}$ and the polar representation for timelike $\mathbb{Q}$ with timelike vector part is

$$\mathbb{Q} = N(\mathbb{Q})(\cos\Phi + \nu\sin\Phi) \tag{1.7}$$

where $\nu = \frac{Bi_+Ci_2+Di_3}{\sqrt{B^2-C^2-D^2}}$ is a unit pure dual split quaternion, $\cos\Phi = \frac{D}{N(\mathbb{Q})}$ and $\sin\Phi = \frac{\sqrt{|B^2-C^2-D^2|}}{N(\mathbb{Q})}$, [16, 17].
Any dual split quaternion can be written in the form $\mathbb{Q} = Ae^{Bi_2}$, where $A = A_0 + A_1i_1$ and $B = A_2 + A_3i_1$ with $A_0, A_1, A_2, A_3$ dual numbers, [18].
The Jacobsthal $J_n$ and Jacobsthal-Lucas $j_n$ number sequences are defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, J_1 = 1, n \geq 2$$
$$j_n = j_{n-1} + 2j_{n-2}, \; j_0 = 2, j_1 = 1, n \geq 2$$

The characteristic equation of these number sequences is $x^2 - x - 2 = 0$, with roots $\alpha = 2$, $\beta = -1$ and corresponding Binet's forms are

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.8}$$

$$j_n = 2^n + (-1)^n \tag{1.9}$$

For more details and relations about these number sequences, [19–22]. The $n^{th}$ dual Jacobsthal $\hat{J}_n$ and Jacobsthal-Lucas $\hat{j}_n$ number sequences are defined by

$$\hat{J}_n = J_n + \varepsilon J_{n+1}$$
$$\hat{j}_n = j_n + \varepsilon j_{n+1}$$

where $J_n$ and $j_n$ are the $n^{th}$ Jacobsthal and Jacobsthal-Lucas numbers, the Binet's formulas for $\hat{J}_n$ and $\hat{j}_n$ are

$$\hat{J}_n = \frac{2^n \underline{\alpha} - (-1)^n \underline{\beta}}{3} \tag{1.10}$$

$$\hat{j}_n = 2^n \underline{\alpha} + (-1)^n \underline{\beta} \tag{1.11}$$

where

$$\underline{\alpha} = 1 + 2\varepsilon$$
$$\underline{\beta} = 1 - \varepsilon$$

Both sequences have applications and appear in various contexts in mathematics, including combinatorial identities, the study of geometric shapes, and connections to the Jacobsthal and Jacobsthal-Lucas. Their properties have been studied in depth, revealing fascinating aspects such as their relationships with binary representations and the golden ratio, though in a manner distinct from the Jacobsthal and Jacobsthal-Lucas. The study of these sequences not only contributes to the field of number theory but also enhances our understanding of the elegant structures and patterns inherent in mathematics.

# 2 Main Results

In this section we defined dual Jacobsthal and dual Jacobsthal-Lucas split quaternions, polar representations and some properties of these split quaternions are discussed.

**Definition 2.1.** The dual Jacobsthal and dual Jacbosthal-Lucas split quaternions are defined as

$$\mathcal{J}_n = \hat{J}_n + \hat{J}_{n+1} i_1 + \hat{J}_{n+2} i_2 + \hat{J}_{n+3} i_3 \tag{2.1}$$

$$\widetilde{\mathcal{J}}_n = \hat{j}_n + \hat{j}_{n+1} i_1 + \hat{j}_{n+2} i_2 + \hat{j}_{n+3} i_3 \tag{2.2}$$

respectively, where $\hat{J}_n$ is $n^{th}$ dual jacobsthal, $\hat{j}_n$ is $n^{th}$ dual jacobsthal-Lucas number and $i_1, i_2, i_3$ follow the rules in Equality (1.1).

From definition, the following recurrence relation can be prove easily

$$\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2},\ n \geq 2$$

and

$$\widetilde{\mathcal{J}}_n = \widetilde{\mathcal{J}}_{n-1} + 2\widetilde{\mathcal{J}}_{n-2},\ n \geq 2$$

**Theorem 2.1.** *The Binet's formulas for dual Jacobsthal and dual Jacobsthal-Lucas split quaternions are*

$$\mathcal{J}_n = \frac{2^n \hat{\alpha} - (-1)^n \hat{\beta}}{3} \tag{2.3}$$

$$\widetilde{\mathcal{J}}_n = 2^n \hat{\alpha} + (-1)^n \hat{\beta} \tag{2.4}$$

*respectively, where*

$$\underline{\alpha} = (1 + 2\varepsilon)(1 + 2i_1 + 4i_2 + 8i_3)$$
$$\underline{\beta} = (1 - \varepsilon)(1 - i_1 + i_2 - i_3)$$

*Proof.* The proof can be done directly by using Equalities (1.10), (1.11), (2.1) and (2.2). □

**Proposition 2.1.** *The norm of dual Jacobsthal and dual Jacobsthal-Lucas split quaternions is*

$$N(\mathcal{J}_n) = \sqrt{(2^{2n+3} + \frac{1}{3}(j_n^2 - 1)) + \varepsilon(2^{2n+5} + j_{2n} + \frac{1}{3}(j_n^2 - 4))}$$
$$N(\widetilde{\mathcal{J}}_n) = 3\sqrt{(2^{2n+3} + 3J_n^2 - \frac{1}{3}) + \varepsilon(2^{2n+5} + 3J_n^2 + j_{2n} - \frac{4}{3})}$$

*where* $J_n$ *and* $j_n$ *are* $n^{th}$ *Jacobsthal and Jacobsthal-Lucas numbers, respectively.*

*Proof.* From Equality (1.2), we have

$$N(\mathcal{J}_n) = \sqrt{|\hat{J}_n^2 + \hat{J}_{n+1}^2 - \hat{J}_{n+2}^2 - \hat{J}_{n+3}^2|}$$

and by using Equality (1.10), we obtain

$$\begin{aligned}\hat{J}_n^2 + \hat{J}_{n+1}^2 - \hat{J}_{n+2}^2 - \hat{J}_{n+3}^2 &= \left(\frac{2^n\underline{\alpha} - (-1)^n\underline{\beta}}{3}\right)^2 + \left(\frac{2^{n+1}\underline{\alpha} - (-1)^{n+1}\underline{\beta}}{3}\right)^2 \\ &- \left(\frac{2^{n+2}\underline{\alpha} - (-1)^{n+2}\underline{\beta}}{3}\right)^2 - \left(\frac{2^{n+3}\underline{\alpha} - (-1)^{n+3}\underline{\beta}}{3}\right)^2 \\ &= -\frac{1}{3}(25 \cdot 2^{2n}(1 + 4\varepsilon) + (-1)^n 2^{n+1}(1 + \varepsilon))\end{aligned}$$

Finally by using Equality (1.8) and doing basic calculations, we get

$$N(\mathcal{J}_n) = \sqrt{(2^{2n+3} + \frac{1}{3}(j_n^2 - 1)) + \varepsilon(2^{2n+5} + j_{2n} + \frac{1}{3}(j_n^2 - 4))}$$

On the other hand for $N(\widetilde{\mathcal{J}}_n)$, by using Equality (1.11) we have

$$\begin{aligned}\hat{j}_n^2 + \hat{j}_{n+1}^2 - \hat{j}_{n+2}^2 - \hat{j}_{n+3}^2 &= \left(2^n\underline{\alpha} + (-1)^n\underline{\beta}\right)^2 + \left(2^{n+1}\underline{\alpha} + (-1)^{n+1}\underline{\beta}\right)^2 \\ &\left(2^{n+2}\underline{\alpha} + (-1)^{n+2}\underline{\beta}\right)^2 - \left(2^{n+3}\underline{\alpha} + (-1)^{n+3}\underline{\beta}\right)^2 \\ &= -3(25 \cdot 2^{2n}(1 + 4\varepsilon) + (-1)^{n+1}2^{n+1}(1 + \varepsilon))\end{aligned}$$

and by doing some necessary calculations, we get

$$N(\widetilde{\mathcal{J}}_n) = 3\sqrt{(2^{2n+3} + 3J_n^2 - \frac{1}{3}) + \varepsilon(2^{2n+5} + 3J_n^2 + j_{2n} - \frac{4}{3})}$$

□

**Corollary 2.2.** *The dual Jacobsthal and dual Jacobsthal-Lucas split quaternions are spacelike split quaternions with spacelike vector part.*

**Theorem 2.3.** *The classical polar representation of dual Jacobsthal split quaternion is*

$$\mathcal{J}_n = N(\mathcal{J}_n)(\sinh\Phi + \nu\cosh\Phi)$$

*where*

$$\nu = \frac{2^{n+1}(1+2\varepsilon)(i_1+2i_2+4i_3)+(-1)^n(1-\varepsilon)(i_1-i_2+i_3)}{3\sqrt{(J_{n+1}^2+2^{2n+3})+\varepsilon(J_{n+1}^2+J_{2n+2}+2^{2n+5})}}$$

*is a pure unit dual split quaternion and* $\Phi$ *is a dual angle such that*

$$\Phi = \tanh^{-1}\left(\frac{J_n}{\sqrt{J_{n+1}^2+2^{2n+3}}}\right)+\varepsilon\frac{j_nJ_{n+1}^2+2^{2n+4}(-1)^n-J_nJ_{2n+2}}{\sqrt{J_{n+1}^2+2^{2n+3}}\,(2^{2n+4}+\frac{2}{3}(j_n^2-1))}$$

*Proof.* Since $\mathcal{J}_n$ is a spacelike dual split quaternion, then by using Equality (1.5), the polar representation is

$$\mathcal{J}_n = N(\mathcal{J}_n)(\sinh\Phi + \nu\cosh\Phi)$$

where

$$\nu = \frac{\hat{J}_{n+1}i_1+\hat{J}_{n+2}i_2+\hat{J}_{n+3}i_3}{\sqrt{-\hat{J}_{n+1}^2+\hat{J}_{n+2}^2+\hat{J}_{n+3}^2}}$$

$$\sinh\Phi = \frac{\hat{J}_n}{N(\mathcal{J}_n)}$$

$$\cosh\Phi = \frac{\sqrt{|\hat{J}_{n+1}^2-\hat{J}_{n+2}^2-\hat{J}_{n+3}^2|}}{N(\mathcal{J}_n)}$$

therefore $\tanh\Phi = \frac{\hat{J}_n}{\sqrt{-\hat{J}_{n+1}^2+\hat{J}_{n+2}^2+\hat{J}_{n+3}^2}}$ and by using Equality (1.10) we get

$$\nu = \frac{2^{n+1}(1+2\varepsilon)(i_1+2i_2+4i_3)-(-1)^n(1-\varepsilon)(i_1-i_2+i_3)}{\sqrt{(J_{n+1}^2+2^{2n+3})+\varepsilon(J_{n+1}^2+2^{2n+5}+J_{2n+2})}}$$

and

$$\tanh\Phi = \frac{J_n+\varepsilon J_{n+1}}{\sqrt{(J_{n+1}^2+2^{2n+3})+\varepsilon(J_{n+1}^2+2^{2n+5}+J_{2n+2})}}$$

Using Equality (1.4) and doing necessary calcualtions we will have

$$\Phi = \tanh^{-1}\left(\frac{J_n}{\sqrt{J_{n+1}^2+2^{2n+3}}}+\varepsilon\frac{j_nJ_{n+1}^2+2^{2n+4}(-1)^n-J_nJ_{2n+2}}{2(J_{n+1}^2+2^{2n+3})^{\frac{3}{2}}}\right)$$

Finally by using Equality (1.3) we obtain

$$\Phi = \tanh^{-1}\left(\frac{J_n}{\sqrt{J_{n+1}^2+2^{2n+3}}}\right)+\varepsilon\frac{j_nJ_{n+1}^2+2^{2n+4}(-1)^n-J_nJ_{2n+2}}{\sqrt{J_{n+1}^2+2^{2n+3}}\,(2^{2n+4}+\frac{2}{3}(j_n^2-1))}$$

□

**Corollary 2.4.** *The classical polar representation of dual Jacobsthal-Lucas split quaternions is*

$$\widetilde{\mathcal{J}}_n = N(\widetilde{\mathcal{J}}_n)(\sinh\hat{\Phi} + \hat{\nu}\cosh\hat{\Phi})$$

*where*

$$\hat{\nu} = \frac{2^{n+1}(1+2\varepsilon)(i_1+2i_2+4i_3)-(-1)^n(1-\varepsilon)(i_1-i_2+i_3)}{3\sqrt{2^{2n+3}+(\frac{j_{n+1}}{3})^2+\varepsilon(2^{2n+5}+J_{2n+2}+(\frac{j_{n+1}}{3})^2)}}$$

*is a pure unit dual split quaternion and* $\hat{\Phi}$ *is a dual angle such that*

$$\hat{\Phi} = \tanh^{-1}\left(\frac{j_n}{\sqrt{j_{n+1}^2+9\cdot 2^{2n+3}}}\right)+\varepsilon\frac{J_n j_{n+1}^2-3\cdot 2^{2n+4}(-1)^n-j_n J_{2n+2}}{\sqrt{j_{n+1}^2+9\cdot 2^{2n+3}}\,(2^{2n+4}+6J_n^2-\frac{2}{3})}$$

*Proof.* The Proof is similar to Theorem (2.3). □

**Proposition 2.2.** *Let* $P = Ai_2 + Bi_3 = (A + Bi_1)i_2$ *be an arbitrary dual split quaternion, if* $P$ *is spacelike then its exponential form is*

$$e^P = \sinh|P| + \frac{A}{|P|}\cosh|P|i_2 + \frac{B}{|P|}\cosh|P|i_3 = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3$$

*and if* $P$ *is timelike, then*

$$e^P = \cosh|P| + \frac{A}{|P|}\sinh|P|i_2 + \frac{B}{|P|}\sinh|P|i_3 = \beta_0 + \beta_2 i_2 + \beta_3 i_3$$

*That is, it is a dual split quaternions which does not contain* $i_1$*'s term.*

*Proof.* Suppose $\mu$ is a spacelike unit dual split quaternion, that is $N(\mu) = 1$, then from Equality (1.5) we have

$$e^{\mu\theta} = \sinh\theta + \mu\cosh\theta$$

if we rewrite $P = |P|\frac{P}{|P|}$, then by taking $\mu = \frac{P}{|P|}$ and $\theta = |P|$ we get the result, we can prove similarly for timelike $P$ by using Equality (1.6). □

Now we give the new polar representations for dual Jacobsthal and dual Jacobsthal-Lucas split quaternions by using Cayley-Dikson's form for split quatenions.

**Theorem 2.5.** *Every dual Jacobsthal split quaterninon* $\mathcal{J}_n = \hat{J}_n + \hat{J}_{n+1}i_1 + \hat{J}_{n+2}i_2 + \hat{J}_{n+3}i_3$ *can be given in the form* $\mathcal{J}_n = \mathbb{A}e^{\mathbb{B}i_2}$*, where* $\mathbb{A}$ *and* $\mathbb{B}$ *are dual Jacobsthal complex numbers, that is*

$$\mathbb{A} = \frac{\hat{J}_n + \hat{J}_{n+1}i_1}{\sqrt{\hat{J}_n^2+\hat{J}_{n+1}^2}}$$

$$\mathbb{B} = \tanh^{-1}\left(\frac{\sqrt{\hat{J}_n^2+\hat{J}_{n+1}^2}}{\sqrt{\hat{J}_{n+2}^2+\hat{J}_{n+3}^2}}\right)\frac{(\hat{J}_n\hat{J}_{n+2}+\hat{J}_{n+1}\hat{J}_{n+3})+(\hat{J}_n\hat{J}_{n+3}-\hat{J}_{n+1}\hat{J}_{n+2})i_1}{\sqrt{(\hat{J}_n^2+\hat{J}_{n+1}^2)(\hat{J}_{n+2}^2+\hat{J}_{n+3}^2)}}$$

*Proof.* Suppose that $\mathbb{A} = a + bi_1$ and $e^{\mathbb{B}i_2} = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3$, then

$$\mathcal{J}_n = \mathbb{A}e^{\mathbb{B}i_2} = a\alpha_0 + b\alpha_0 i_1 + (a\alpha_2 - b\alpha_3)i_2 + (a\alpha_3 + b\alpha_2)i_3$$

if $\alpha_0 = 0$, then we can select $a = 1$ and $b = 0$, we will get $\mathbb{A} = 1$. For $\alpha_0 \neq 0$, we construct a complex number $\psi = a\alpha_0 + b\alpha_0 i_1 = \hat{J}_n + \hat{J}_{n+1}i_1$ and then $\mathbb{A} = \frac{\psi}{|\psi|}$, by using Equality (1.10) the explicit form of $\mathbb{A}$ is

$$\mathbb{A} = \frac{2^n(1+2\varepsilon)(1+2i_1)+(-1)^{n+1}(1-\varepsilon)(1-i_1)}{\sqrt{j_n^2+2^{2n+2}+\varepsilon(j_n^2+9J_{2n}+2^{2n+4})}}$$

Since $\mathbb{A}$ is a unit dual complex number then $\mathbb{A}^{-1} = \overline{\mathbb{A}} = \frac{\hat{J}_n - \hat{J}_{n+1} i_1}{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}$, where $\overline{\mathbb{A}}$ is conjugate of $\mathbb{A}$ and

$$e^{\mathbb{B} i_2} = \overline{\mathbb{A}} \mathcal{J}_n = \frac{(\hat{J}_n^2 + \hat{J}_{n+1}^2) + (\hat{J}_n \hat{J}_{n+2} + \hat{J}_{n+1} \hat{J}_{n+3}) i_2 + (\hat{J}_n \hat{J}_{n+3} - \hat{J}_{n+1} \hat{J}_{n+2}) i_3}{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}$$

The norm of $e^{\mathbb{B} i_2}$ is

$$|e^{\mathbb{B} i_2}| = \sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2 - \hat{J}_n^2 - \hat{J}_{n+1}^2}$$

Since $\frac{e^{\mathbb{B} i_2}}{|e^{\mathbb{B} i_2}|}$ is a unit spacelike dual split quaternion, then its classical polar form is

$$\frac{e^{\mathbb{B} i_2}}{|e^{\mathbb{B} i_2}|} = \sinh \hat{\Phi} + \hat{\mu} \cosh \hat{\Phi}$$

where $\hat{\Phi} = \phi + \varepsilon \phi^*$ is dual angle, then we can write

$$\sinh \hat{\Phi} = \frac{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2 - \hat{J}_n^2 - \hat{J}_{n+1}^2}}$$

$$\cosh \hat{\Phi} = \frac{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2 - \hat{J}_n^2 - \hat{J}_{n+1}^2}}$$

$$\hat{\mu} = \frac{(\hat{J}_n \hat{J}_{n+2} + \hat{J}_{n+1} \hat{J}_{n+3}) i_2 + (\hat{J}_n \hat{J}_{n+3} - \hat{J}_{n+1} \hat{J}_{n+2}) i_3}{\sqrt{(\hat{J}_n^2 + \hat{J}_{n+1}^2)(\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2)}}$$

which gives

$$\tanh \hat{\Phi} = \frac{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}}$$

Finally from $\mathbb{B} i_2 = \hat{\mu} \hat{\Phi}$ we get the result and by using Equality (1.10) the explicit form of $\mathbb{B}$ can be write easily. □

**Corollary 2.6.** *Every dual Jacobsthal-Lucas split quaterninon* $\widetilde{\mathcal{J}}_n = \hat{j}_n + \hat{j}_{n+1} i_1 + \hat{j}_{n+2} i_2 + \hat{j}_{n+3} i_3$ *can be given in the form* $\widetilde{\mathcal{J}}_n = \mathbb{A} e^{\mathbb{B} i_2}$*, where* $\mathbb{A}$ *and* $\mathbb{B}$ *are dual Jacobsthal-Lucas complex numbers, that is*

$$\mathbb{A} = \frac{\hat{j}_n + \hat{j}_{n+1} i_1}{\sqrt{\hat{j}_n^2 + \hat{j}_{n+1}^2}}$$

$$\mathbb{B} = \tanh^{-1} \left( \frac{\sqrt{\hat{j}_n^2 + \hat{j}_{n+1}^2}}{\sqrt{\hat{j}_{n+2}^2 + \hat{j}_{n+3}^2}} \right) \frac{(\hat{j}_n \hat{j}_{n+2} + \hat{j}_{n+1} \hat{j}_{n+3}) + (\hat{j}_n \hat{j}_{n+3} - \hat{j}_{n+1} \hat{j}_{n+2}) i_1}{\sqrt{(\hat{j}_n^2 + \hat{j}_{n+1}^2)(\hat{j}_{n+2}^2 + \hat{j}_{n+3}^2)}}$$

*Proof.* The proof can be done similar to Theorem (2.5). □

**Example 2.7.** *Find the new polar representation for* $\mathcal{J}_1 = (1 + \varepsilon) + (1 + 3\varepsilon) i_1 + (3 + 5\varepsilon) i_2 + (5 + 11\varepsilon) i_3$*. We have* $\mathcal{J}_1 = \mathbb{A} e^{\mathbb{B} i_2}$*, where*

$$\mathbb{A} = \frac{\sqrt{2}}{2} ((1 - \varepsilon) + (1 + \varepsilon) i_1)$$

*and*

$$\mathbb{B} = \frac{1}{\sqrt{17}} \tanh^{-1} \left( \frac{1}{\sqrt{17}} + \varepsilon \frac{\sqrt{17} - 35\sqrt{2}}{17} \right) \left( (4 + \frac{13}{17} \varepsilon) + (1 - \frac{52}{17} \varepsilon) i_1 \right)$$

**Theorem 2.8** (Catalan's identities). *For positive integers $n$ and $r$ with $n \geq r \geq 1$, we have*

$$\mathcal{J}_{n+r}\mathcal{J}_{n-r} - \mathcal{J}_n^2 = \frac{1}{3}(-1)^{n-r}2^{n-r}J_r(1+\varepsilon)\left(j_r(1-13i_1+i_2-13i_3)+2(-1)^r(-1+i_1+5i_2+7i_3)\right)$$
$$\widetilde{\mathcal{J}}_{n+r}\widetilde{\mathcal{J}}_{n-r} - \widetilde{\mathcal{J}}_n^2 = 3(-1)^{n-r}2^{n-r}J_r(1+\varepsilon)\left(j_r(-1+13i_1-i_2+13i_3)-2(-1)^r(-1+i_1+5i_2+7i_3)\right)$$

where $J_n$ and $j_n$ are $n^{th}$ jacobsthal and Jacobsthal-Lucas numbers.

*Proof.* By using Equality (2.3), we get

$$\begin{aligned}
\mathcal{J}_{n+r}\mathcal{J}_{n-r} - \mathcal{J}_n^2 &= \frac{1}{3}\left(2^{n+r}\underline{\alpha}-(-1)^{n+r}\underline{\beta}\right)\frac{1}{3}\left(2^{n-r}\underline{\alpha}-(-1)^{n-r}\underline{\beta}\right)\\
&- \frac{1}{3}\left(2^{n}\underline{\alpha}-(-1)^{n}\underline{\beta}\right)\frac{1}{3}\left(2^{n}\underline{\alpha}-(-1)^{n}\underline{\beta}\right)\\
&= \frac{1}{9}2^n(-1)^n\left(\underline{\alpha\beta}\left(1-2^r(-1)^{-r}\right)+\underline{\beta\alpha}\left(1-2^{-r}(-1)^r\right)\right)\\
&= \frac{1}{3}2^n(-1)^nJ_r\left((-1)^{r+1}\underline{\alpha\beta}+2^{-r}\underline{\beta\alpha}\right)\\
&= \frac{1}{3}2^{n-r}(-1)^{n-r}J_r\left((-1)^r\underline{\beta\alpha}-2^r\underline{\alpha\beta}\right)
\end{aligned}$$

from using

$$\underline{\alpha\beta} = (1+\varepsilon)(-1+13i_1-i_2+13i_3) \tag{2.5}$$
$$\underline{\beta\alpha} = (1+\varepsilon)(-1-11i_1+11i_2+i_3) \tag{2.6}$$

we get the result. Similarly, the proof for Jacobsthal-Lucas can be done. □

**Corollary 2.9** (Cassini's identities). *For positive integer $n$, the following identities hold*

$$\mathcal{J}_{n+1}\mathcal{J}_{n-1} - \mathcal{J}_n^2 = (-1)^n2^{n-1}(1+\varepsilon)\left(-1+5i_1+3i_2+9i_3\right)$$
$$\widetilde{\mathcal{J}}_{n+1}\widetilde{\mathcal{J}}_{n-1} - \widetilde{\mathcal{J}}_n^2 = 9(-1)^{n-1}2^{n-1}(1+\varepsilon)\left(-1+5i_1+3i_2+9i_3\right)$$

*Proof.* The proof can be done by taking $r = 1$ in Theorem (2.8). □

**Theorem 2.10** (d'Ocagne's identities). *For positive integers $n$ and $m$ with $n \geq m$ we have*

$$\mathcal{J}_{m+1}\mathcal{J}_n - \mathcal{J}_m\mathcal{J}_{n+1} = \frac{1+\varepsilon}{3}\left(2^n(-1)^m(-1-11i_1+11i_2+i_3)-2^m(-1)^n(-1+13i_1-i_2+13i_3)\right)$$
$$\widetilde{\mathcal{J}}_{m+1}\widetilde{\mathcal{J}}_n - \widetilde{\mathcal{J}}_m\widetilde{\mathcal{J}}_{n+1} = 3(1+\varepsilon)\left(2^m(-1)^n(-1+13i_1-i_2+13i_3)-2^n(-1)^m(-1-11i_1+11i_2+i_3)\right)$$

*Proof.* By using Equality (2.3) we have

$$\begin{aligned}
\mathcal{J}_{m+1}\mathcal{J}_n - \mathcal{J}_m\mathcal{J}_{n+1} &= \frac{1}{3}\left(2^{m+1}\underline{\alpha}-(-1)^{m+1}\underline{\beta}\right)\frac{1}{3}\left(2^{n}\underline{\alpha}-(-1)^{n}\underline{\beta}\right)\\
&- \frac{1}{3}\left(2^{m}\hat{\alpha}-(-1)^{m}\underline{\beta}\right)\frac{1}{3}\left(2^{n+1}\underline{\alpha}-(-1)^{n+1}\underline{\beta}\right)\\
&= \frac{1}{9}\left(-\underline{\alpha\beta}\,2^{m+1}(-1)^n-\underline{\beta\alpha}\,(-1)^{m+1}2^n+\underline{\alpha\beta}\,(-1)^{n+1}2^m+\underline{\beta\alpha}\,2^{n+1}(-1)^m\right)\\
&= \frac{1}{3}\left(\underline{\beta\alpha}\,2^n(-1)^m-\underline{\alpha\beta}\,2^m(-1)^n\right)
\end{aligned}$$

Finally by using Equalities (2.5) and (2.6) the result is clear. The proof for Jacobthal-Lucas can be done similarly by using Equalities (2.4), (2.5) and (2.6). □

# 3 Conclusions

In this paper, we define dual Jacobstahl and dual Jacobsthal-Lucas split quaternions. The polar representations of these split quaternions have been obtained similar to the real quaternions. For this, the modulus and argument have been calculated from an arbitrary split quaternion. Our work opens up a plethora of avenues for future research. One particularly promising direction is the application of these concepts to dual split quaternions, incorporating the sequences of Jacobsthal and Jacobsthal-Lucas numbers. This approach is not just a mere extension of our current framework but also a potential pathway to uncover new relationships and properties within the realm of quaternionic analysis. In conclusion, our study not only adds a new chapter to the field of quaternionic analysis but also sets the stage for a deeper and more comprehensive exploration of mathematical structures that intertwine number sequences with quaternionic frameworks. As we continue to unravel the mysteries of these fascinating mathematical entities, we anticipate that our work will inspire further research and collaboration in this dynamic and ever-evolving field.

# Competing Interests

The authors contributed to the writing of this paper. The author read and approved the final manuscript. The authors declares no conflict of interest.

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