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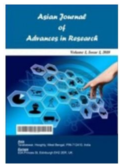

NONLOCAL INITIAL VALUE PROBLEMS FOR HILFER-TYPE FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS WITH IMPULSE

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ABSTRACT

In this note, we verify the existence of solutions to nonlocal initial value problems for Hilfer-type fractional hybrid differential equations with impulsive condition. Then, we use prerequisites of Hilfer fractional calculus and the standard fixed point theorem due to Dhage for deriving the existence results in the weighted space of continuous functions. An example is presented to illustrate the theory results.

Keywords: Hilfer fractional derivative; Existence; Fixed point; Fractional hybrid differential equations.

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1 INTRODUCTION

Fractional differential equations (FDEs) have recently proved to be valuable tools in the modeling of many phenomena. As a result of its wide applicability in biology, medicine and in more and more fields, the theory of FDEs has recently been attracting increasing attention, see the monographs of Hilfer [1], Kilbas [2] and Podlubny [3]. Applied problems require definitions of fractional derivatives allowing the use of physically interpretable initial conditions and boundary conditions. One more attractive class of problems involves fractional hybrid differential equations. For some work on this topic, one can refer to [4, 5, 6, 7]. In [1], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see

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also [8, 9]). In the recent years, some authors have considered Hilfer fractional derivative, see [10, 11, 12, 13, 8, 14, 15, 16, 17] and references therein.

This note deals with the existence of solutions for the nonlocal initial value problems (IVPs), for Hilfer-type fractional hybrid differential equations (FHDEs) with impulsive effects

$$D_{0^+}^{\alpha,\beta}\left(\frac{U(t)}{F(t,U(t))}\right) = G(t,U(t)), \quad t \in J := [0,T], \tag{1.1}$$

$$U(t_k^+) = U(t_k^-) + V_k, \quad k = 1,2,...,m, \ V_k \in R \tag{1.2}$$

$$I_{0^+}^{1-\gamma}U(0) + \eta(U) = U_0, \tag{1.3}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $F \in C(J \times R, R|\{0\})$, $G : C(J \times R, R)$ and $\eta : C(C,J) \times R$, $I_{0^+}^{1-\gamma}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma$ and $U_0 \in R$. t_k satisfies $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $U(t_k^+) = \lim_{\epsilon\to 0^+} U(t_k + \epsilon)$ and $U(t_k^-) = \lim_{\epsilon\to 0^-} U(t_k + \epsilon)$ represents the right and left limits of $U(t)$ at $t = t_k$.

Impulsive differential equations (IDEs) have become essential in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of IDEs with fixed moments; see for instance the monographs by Bainov and Simeonov [18], Benchohra et al. [19] and Lakshmikantham et al. [20] and the references therein. Particular attention has been given to differential equations at variable moments of impulse; see for instance the papers by Bajo and Liz [21].

2 PREREQUISITES

Let $C(J,R)$ denotes the Banach space of all continuous real-valued functions defined on J with the norm

$$\|U\| = \sup\{|U(t)| : \ t \in J\}.$$

For $t \in J$, we define $U_r(t) = t^r U(t)$, $r \geq 0$. Let $C_r(J,R)$ be the space of all continuous functions U such that $U_r \in C(J,R)$ which is indeed a Banach space endowed with the norm

$$\|U\|_C = \sup\{t^r|U(t)| : \ t \in J\}.$$

Let $0 \leq \gamma \leq 1$ and $C_\gamma(J,R)$ denote the weighted space of continuous function defined by

$$C(J,R) = \left\{G(t) : \ t^\gamma G(t) \in C(J,R), \ \|V\|_{C_\gamma} = \|t^\gamma G(t)\|_C\right\}.$$

In the following we denote $\|V\|_{C_\gamma}$ by $\|V\|_C$.

Definition 2.1. The fractional integral operator of order $\alpha > 0$ for a fractional function F cam be defined as

$$I^\alpha F(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{F(s)}{(t-s)^{\alpha-1}}ds, t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. [1](Hilfer derivative). Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $F \in L^1(J)$, $I_{0^+}^{(1-\alpha)(1-\beta)} \in C_\gamma^1[J,R]$. The Hilfer fractional derivative of order α and type β of F is defined as

$$(D_{0^+}^{\alpha,\beta}F)(t) = \left(I_{0^+}^{\beta(1-\alpha)}\frac{d}{dt}I_{0^+}^{(1-\alpha)(1-\beta)}F\right)(t); \quad \text{for a.e. } t \in J. \tag{2.1}$$

Lemma 2.1. *[11] Let $F : J \times R \to R$ be a function such that $F(\cdot, U(\cdot)) \in C_\gamma(J, R)$ for any $U \in C_\gamma(J, R)$. A function $U \in C_\gamma(J, R)$ is a solution of fractional initial value problem:*

$$\begin{cases} D_{0^+}^{\alpha,\beta} U(t) = F(t, U(t)), \ 0 < \alpha < 1, \ 0 \le \beta \le 1, \\ I_{0^+}^{1-\gamma} U(0) = U_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

if and only if U satisfies the following Volterra integral equation:

$$U(t) = \frac{U_0 t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, U(s)) ds.$$

Further details can be found in [22]. From Lemma 2.1 we have the following result.

Lemma 2.2. *[23] Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 \le \beta \le 1$. Let $F : J \times R \to R$ be a function such that $F \in C_\gamma(J, R)$ for any $U \in C_\gamma(J, R)$. If $G \in C_\gamma(J, R)$, then U satisfies*

$$D_{0^+}^{\alpha,\beta} \left(\frac{U(t)}{F(t, U(t))} \right) = G(t, U(t)), \quad t \in J := [0, T], \tag{2.2}$$

$$I_{0^+}^{1-\gamma} U(0) = \phi, \tag{2.3}$$

if and only if U satisfies the integral equation

$$U(t) = F(t, U(t)) \left(\frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s, U(s)) ds \right), \quad t > 0. \tag{2.4}$$

We adopt some ideas from [24].

Lemma 2.3. *Given $V \in C(J, R)$, the integral solution of IVP*

$$D_{0^+}^{\alpha,\beta} \left(\frac{U(t)}{F(t, U(t))} \right) = V(t), \ 0 < t < 1, \tag{2.5}$$

$$U(t_k^+) = U(t_k^-) + V_k, \ k = 1, 2, ...m, \ V_k \in R, \tag{2.6}$$

$$I_{0^+}^{1-\gamma} U(0) + \eta(U) = U_0, \tag{2.7}$$

is given by

$$U(t) = \begin{cases} F(t, U(t)) \left(\frac{U_0 - \eta(U)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V(s) ds \right), & \text{for } t \in (0, t_1], \\ F(t, U(t)) \left(\frac{U_0 - \eta(U) + V_1}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V(s) ds \right), & \text{for } t \in (t_1, t_2], \\ F(t, U(t)) \left(\frac{U_0 - \eta(U) + V_1 + V_2}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V(s) ds \right), & \text{for } t \in (t_2, t_3], \\ \cdots \\ F(t, U(t)) \left(\frac{U_0 - \eta(U) + \sum_{i=0}^m V_i}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V(s) ds \right), & \text{for } t \in (t_m, T]. \end{cases} \tag{2.8}$$

Proof. Assume that U satisfies equation (1.1)-(1.3). If $t \in (0, t_1]$, then $D_{0^+}^{\alpha,\beta} \left(\frac{U(t)}{F(t,U(t))} \right) = V(t)$, $t \in (0, t_1]$ with $I_{0^+}^{1-\gamma} U(0) + \eta(U) = U_0$. By virtue of Lemma 2.2, one can obtain

$$U(t) = F(t, U(t)) \left(\frac{U_0 - \eta(U)}{\Gamma(\gamma)} t^{\gamma-1} + + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V(s) ds \right), \ \text{for } t \in (0, t_1].$$

If $(t_1, t_2]$ then $D_{0^+}^{\alpha,\beta} \left(\frac{U(t)}{F(t,U(t))} \right) = V(t)$, $t \in (t_1, t_2]$ with $U(t_1^+) = U(t_1^-) + V_1$. Then we have

$$U(t_1^+) = F(t, U(t)) \left(U_0 - \eta(U) + \int_0^{t_1} V(s) ds + V_1 \right).$$

By Lemma 2.2, we get

$$\begin{aligned}U(t) &= F(t,U(t))\left(\frac{U(t_1^+)-\eta(U)}{\Gamma(\gamma)}t^{\gamma-1}+\frac{t^{\gamma-1}}{\Gamma(\gamma)}\int_0^{t_1}V(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right)\\ &= F(t,U(t))\left(\frac{U_0-\eta(U)}{\Gamma(\gamma)}t^{\gamma-1}+\frac{V_1}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right)\\ &= F(t,U(t))\left(\frac{U_0-\eta(U)+V_1}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right),\ t\in(t_1,t_2]\end{aligned}$$

without loss of generality, for $t\in(t_i,t_{i+1}]$, $i=1,2,...m$, we get

$$U(t)=F(t,U(t))\left(\frac{U_0-\eta(U)+\sum_{i=1}^m V_i}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right),\ t\in(t_i,t_{i+1}].$$

On the other hand, assume that U satisfies the integral equations (2.8). If $t\in(0,t_1]$, then $I_{0^+}^{1-\gamma}U(0)+\eta(U)=U_0$, we get $D_{0^+}^{\alpha,\beta}\left(\frac{U(t)}{F(t,U(t))}\right)=V(t)$. Similarly, if $t\in(t_i,t_{i+1}]$, we obtain $D_{0^+}^{\alpha,\beta}\left(\frac{U(t)}{F(t,U(t))}\right)=V(t)$ and $U(t_k^+)=U(t_k^-)+V_k$, $k=1,2,...,m$. This completes the proof. $\square$

Theorem 2.4. *[25, 26] Let S be a non-empty, closed convex and bounded subset of the Banach algebra R, let $A:R\to R$ and $B:S\to R$ be two operators such that:*

(a) *A is Lipschitzian with a Lipschitz constant k;*

(b) *B is completely continuous;*

(c) *$U=AUBV \implies U\in S$ for all $V\in S$, and*

(d) *$Mk<1$, where $M=\|B(S)\|=\sup\{\|B(U)\|:\ U\in S\}$.*

Then the operator equation $U=AUBU$ has a solution.

3 EXISTENCE RESULTS

We introduce the following hypotheses:

(H1) The function $F:J\times R\to R|\{0\}$ is bounded continuous and there exists a positive bounded function ϕ with bound $\|\phi\|$ such that

$$|F(t,U(t))-F(t,V(t))|\le\phi(t)\,|U(t)-V(t)|,$$

for $t\in J$ and for all $U,V\in R$.

(H2) There exists a function $P\in C(J,R^+)$ and a continuous nondecreasing function $\Omega:[0,\infty)\to(0,\infty)$ such that

$$|G(t,U(t))|\le P(t)\Omega(|U|),\quad (t,U)\in J\times R.$$

(H3) There exists a number $r>0$ such that

$$r\ge K\left[\sum_{i=0}^m|V_i|+\frac{|U_0+\mathfrak{G}|}{\Gamma(\gamma)}+\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\|P\|\,\Omega(r)\right],\tag{3.1}$$

where $|F(t,U)|\le K,\ \forall\ (t,U)\in[0,T]\times R$ and

$$\|\phi\|\left[\sum_{i=0}^m|V_i|+\frac{|U_0|+\mathfrak{G}}{\Gamma(\gamma)}+\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\|P\|\,\Omega(r)\right]<1.$$

Theorem 3.1. *Assume that (H1)-(H3) are satisfied. Then the problem* (1.1)-(1.3) *has at least one solution on* J*.*

Proof. Set $X = C(J, R)$ and define a subset S of X as

$$S = \left\{U \in X : \ \|U\|_C \leq r\right\},$$

where r satisfies inequality (3.1).
Clearly, S is closed, convex and bounded subset of the banach space X and $G = \sup_{U\times X} |\eta(U)|$. By Lemma 2.3, the IVP (1.1)-(1.3) is equivalent to the integral equation

$$U(t) = \begin{cases} F(t,U(t))\left(\frac{U_0-\eta(U)}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G(s,U(s))ds\right), & \text{for } t \in (0,t_1], \\ F(t,U(t))\left(\frac{U_0-\eta(U)+V_1}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G(s,U(s))ds\right), & \text{for } t \in (t_1,t_2], \\ F(t,U(t))\left(\frac{U_0-\eta(U)+V_1+V_2}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G(s,U(s))ds\right), & \text{for } t \in (t_2,t_3], \\ \cdots \\ F(t,U(t))\left(\frac{U_0-\eta(U)+\sum_{i=0}^m V_i}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G(s,U(s))ds\right), & \text{for } t \in (t_m,T]. \end{cases} \tag{3.2}$$

Define the operators $A : X \to X$ and $B : S \to X$

$$AU(t) = F(t, U(t)), \quad t \in (t_m, T], \tag{3.3}$$

$$BU(t) = \frac{U_0 - \eta(U) + \sum_{i=0}^m V_i}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G(s,U(s))ds. \tag{3.4}$$

Then, $U = AUBU$. We shall show that the operators A and B satisfy all the hypotheses of Theorem 2.4. For the sake of clarity, we split the proof as follow:
Claim 1.

We first show that A is a Lipschitz on X, (i.e.) (a) of Theorem 2.4 holds.

Let $U, V \in X$. Then by (H1), we have

$$\begin{aligned} \left|t^{1-\gamma}\left(AU(t) - AV(t)\right)\right| &= t^{1-\gamma}\left|F(t,U(t) - F(t,V(t))\right| \\ &\leq \phi(t)t^{1-\gamma}\left|U(t) - V(t)\right| \\ &\leq \|\phi(t)\|\,\|U-V\|_C, \quad \forall\ (t_m,T]. \end{aligned}$$

Taking the supremum over the interval $(t_m, T]$, we get

$$\|AU - AV\|_C \leq \|\phi\|\,\|U-V\|_C, \quad \forall\ U,V \in R.$$

So A is a Lipschitz on X with Lipschitz constant $\|\phi\|$.
Claim 2.

The operator B is completely continuous on S, i.e. (b) of Theorem 2.4 holds.

First we show that B is continuous on S.

Let $\{U_n\}$ be a sequence of S converging to a point $U \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} t^{1-\gamma}BU_n(t) &= \lim_{n\to\infty}\left(\frac{U_0-\eta(U)+\sum_{i=0}^\infty V_i}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t-s)^{\alpha-1}G(s,U_n(s))ds\right) \\ &= \left(\frac{U_0-\eta(U)+\sum_{i=0}^\infty V_i}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t-s)^{\alpha-1}\lim_{n\to\infty} G(s,U_n(s))ds\right) \\ &= \left(\frac{U_0-\eta(U)+\sum_{i=0}^\infty V_i}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t-s)^{\alpha-1}G(s,U(s))ds\right) \\ &= t^{1-\gamma}BU(t), \quad \forall \quad t \in (t_m,T]. \end{aligned}$$

This shows that B is continuous on S. It is sufficient to show that $B(s)$ is a uniformly bounded and equicontinuous set in X.

$$\begin{aligned} t^{1-\gamma}\left|BU(t)\right| &= \left|\frac{U_0-\eta(U)+\sum_{i=0}^{\infty}V_i}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}G(s,U(s))ds\right| \\ &= \left[\frac{|U_0|-\mathfrak{G}+\sum_{i=0}^{\infty}|V_i|}{\Gamma(\gamma)}+\|P\|\,\Omega(r)\frac{T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\right] \\ &= \frac{|U_0|-\mathfrak{G}+\sum_{i=0}^{\infty}|V_i|}{\Gamma(\gamma)}+\|P\|\,\Omega(r)\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}, \quad \forall \quad t\in(t_m,T]. \end{aligned}$$

Taking supremum over the interval $(t_m,T]$, then we have

$$\|B(U)\|_C \le \frac{|U_0|-\mathfrak{G}+\sum_{i=0}^{\infty}|V_i|}{\Gamma(\gamma)}+\|P\|\,\Omega(r)\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}, \quad \forall\, U\in S.$$

This shows that B is unifromly bounded on S.

Next we show that B is an equicontinuous set in X. Let $t_1,t_2\in(t_m,T]$ with $t_1<t_2$ and $U\in S$. Then we have

$$\begin{aligned} &\left|t_2^{1-\gamma}(BU)(t_2)-t_1^{1-\gamma}(BU)(t_1)\right| \\ &\le \frac{\|P\|\,\Omega(r)+\sum_{i=0}^{m}V_i}{\Gamma(\alpha)}\left|\int_0^{t_2}t_2^{1-\gamma}(t_2-s)^{\alpha-1}ds-\int_0^{t_1}t_1^{1-\gamma}(t_1-s)^{\alpha-1}ds\right| \\ &\le \frac{\|P\|\,\Omega(r)+\sum_{i=0}^{m}V_i}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[t_2^{1-\gamma}(t_2-s)^{\alpha}-t_1(t_1-s)^{\alpha-1}\right]ds\right| \\ &\quad+\frac{\|P\|\,\Omega(r)}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}t_2^{1-\gamma}(t_2-s)^{\alpha-1}ds\right|. \end{aligned}$$

Obviously, the right-side of the above inequality tends to zero independently of $U\in S$ as $t_2-t_1\to 0$. Therefore, it follows from the Arzela-Ascoli theorem that B is a completely continuos operator on S.

Claim 3.

Next, we show that hypothesis (c) of Theorem 2.4 is satisfied.

Let $U\in X$ and $V\in S$ be arbitrary elements such that $U=AUBV$. Then we have

$$\begin{aligned} t^{1-\gamma}\left|U(t)\right| &= t^{1-\gamma}\left|AU(t)\right|\left|BV(t)\right| \\ &= |F(t,U(t))|\left(\left|\frac{U_0-\eta(U)+\sum_{i=1}^{m}V_i}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right|\right) \\ &\le K\left|\frac{U_0-\eta(U)+\sum_{i=1}^{m}V_i}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right| \\ &\le K\left[\frac{|U_0|+\mathfrak{G}+\sum_{i=1}^{m}|V_i|}{\Gamma(\gamma)}+\|P\|\,\Omega(r)\frac{T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}V(s)ds\right] \\ &\le K\left[\frac{|U_0|+\mathfrak{G}+\sum_{i=1}^{m}|V_i|}{\Gamma(\gamma)}+\|P\|\,\Omega(r)\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right]. \end{aligned}$$

Taking supremum for $t\in(t_m,T]$, we obtain

$$\|U\|\le K\left[\frac{|U_0|+\mathfrak{G}+\sum_{i=1}^{m}|V_i|}{\Gamma(\gamma)}+\|P\|\,\Omega(r)\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right]\le r,$$

that is $U=S$.

Claim 4. Now we show that $Mk < 1$, that is, (d) of Theorem 2.4 holds. This is obtain by (H4), since we have

$$\begin{aligned} M &= \|\mathfrak{B}(S)\| = \sup\{\|BU\| : U \in S\} \\ &\leq \frac{|U_0| + \mathfrak{G} + \sum_{i=1}^{m}|V_i|}{\Gamma(\gamma)} + \|P\|\,\Omega(r)\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} \end{aligned}$$

and $k = \|\phi\|$.

Thus all the condition of Theorem 2.4 are satisfied and hence operator equation $U = AUBU$ has a solution in S. In consequence, the problem (1.1)-(1.3) has a solution on $(t_m, T]$. This completes the proof. □

4 AN EXAMPLE

Consider the problem

$$D^{\frac{1}{2},\frac{1}{2}}\left(\frac{U(t)}{F(t,U(t))}\right) = G(t,U(t)), \quad t \in [0,1], \tag{4.1}$$

$$U(t_k^+) = U(t_k^-) + \frac{1}{4}, \tag{4.2}$$

$$I^{1-\gamma}U(0) + \sum_{i=1}^{m} c_i U(t_i) = 1, \tag{4.3}$$

where $0 < t_1 < t_2 < ... < t_m < 1$, $c_i = 1, ..., m$ are positive constants with $\sum_{i=1}^{m} c_i \leq \frac{1}{3}$.
Here,

$$\begin{aligned} F(t,U) &= \frac{1}{5}\left(\sin t \tan^{-1} U + \frac{\pi}{2}\right), \\ G(t,U) &= \frac{1}{10}\left(\frac{1}{6}|U| + \frac{1}{8}\cos U + \frac{|U|}{4(1+|U|)} + \frac{1}{16}\right). \end{aligned}$$

Obviously, $|F(t,U)| \leq \frac{\sqrt{\pi}}{5} = K$, $\|\phi\| = \frac{\sqrt{\pi}}{5}$ and $|G(t,U)| \leq \frac{1}{10}\left(\frac{1}{6}|U| + \frac{7}{16}\right)$. We choose $\|P\| = \frac{1}{10}$, $\Omega(r) = \frac{1}{6}r + \frac{7}{16}$. Clearly, all the condition of Theorem 3.1 are satisfied. Hence by the conclusion of Theorem 3.1, it follows that problem (4.1)-(4.3) has a solution.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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