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An Orthogonal Polynomial Based Iterative Procedure for Finding the Root of the Equation $f(x) = 0$

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

Iterative methods provide hope for many nonlinear engineering problems that cannot be solved through analytic procedures. In this article, orthogonal polynomial based iterative schemes are developed for the approximate solutions of nonlinear algebraic and transcendental equations. Basically, Mamadu-Njoseh orthogonal polynomials are employed as basis functions to derive the new iterative schemes called the "Mamadu Δ^2 and Δ^3 iterative schemes". Convergence analysis of the schemes shows the convergence rate as of order 3 and 4, respectively. Numerical experimental of the new schemes show the feasibility and correctness of the method.

Keywords: *Orthogonal polynomials; Mamadu-Njoseh polynomials; convergence; Newton-Raphson scheme; algebraic and transcendental equations.*

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1 Introduction

Nonlinear equations are frequently encountered in many engineering problems. An N coupled nonlinear equations in N variables, $x_1, x_2, \ldots, x_n$, has the form [1,2]

$$\begin{aligned} G_1(x_1) &= 0 \\ G_2(x_2) &= 0 \\ &\vdots \\ G_N(x_n) &= 0, \end{aligned} \tag{1.1}$$

Equation(1.1) can be rewritten as

$$G(x) = \begin{pmatrix} G_1(x_1) \\ G_2(x_2) \\ G_3(x_3) \\ \vdots \\ G_N(x_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.2}$$

More than one solution of x can exist that may satisfy (1.2). A typical example of (1.2) is the nonlinear characteristics equation of Nth degree [3]. When $N = 1$ with the variable λ, we obtain N ordinary differential equations, which is relevant to the study of system analysis. Again, the simulation of chemical plants in its steady state can be modeled by thousands of coupled nonlinear equations involving several variables [4, 5].

A lot of constraints are encountered in solving (1.2) analytically, which may be due to perturbation, making of weak assumptions, linearization or quasi linearization, and among others. Consequently, iterative schemes have been adopted by various mathematicians as means of solving these equations. Some popular iterative techniques over the years include, the bisection method, Newton-Raphson method, fixed point iteration method, successive substitution method, Muller's method, Secant method, and among others. Amongst others, the Newton-Raphson is quite easy to implement and converges fast but requires more computational stress [6–9].

This paper offers to derive and implement new iterative scheme that performs better than other methods, especially the Newton-Raphson method. The new scheme is orthogonal polynomials based, that is, applying Mamadu-Njoseh polynomials [10-17] as basis functions as series solutions subjects to three relevant conditions. The class of algebraic and transcendental equations and possibly any nonlinear equations are employed to test the rate of convergence, accuracy and effectiveness of the new scheme. For future references, the new iterative schemes shall be called "Mamadu Δ^2 and Δ^3 iterative methods".

2 Derivation of Mamadu Δ^2 And Δ^3 Iterative Schemes

In this section, we shall derive new iterative formulas for the approximate solutions of both algebraic and transcendental equations that depend on the degree of orthogonal polynomials. Basically, we derive here the iterative formulas with orthogonal polynomials of degrees two and three, respectively.

Now, let $g(x)$ define an orthogonal polynomial of the form

$$g(x) = \sum_{i=0}^{n} a_i \varphi_i(x), a_0 \neq 0, \tag{2.1}$$

where a_i's are constants and $\varphi_i(x), i = 0,1,2,\ldots,n$, are Mamadu-Njoseh polynomials defined in $[-1, 1]$ with respect to the weight function $w(x) = 1 + x^2$. For $n = 2$ in (2.1), we obtain a second degree polynomial of the form

$$g(x) = a_0 + a_1 x + \frac{a_2}{3}(5x^2 - 2) = 0 \tag{2.2}$$

To estimate a_0, a_1, a_2 in (2.2), we will prescribe three conditions on $g(x)$ as

$$g_k = a_0 + a_1 x_k + \frac{a_2}{3}(5x_k^2 - 2) \quad (2.3)$$

$$g_k' = a_1 + \frac{10x_k a_2}{3} \quad (2.4)$$

$$g_k'' = \frac{10a_2}{3} \quad (2.5)$$

Solving (2.3) – (2.5) we obtain,

$$\left.\begin{array}{c} a_0 = \frac{1}{2}g_k''x_k^2 - g_k'x_k + g_k + \frac{1}{5}g_k'' \\ a_1 = g_k''x_k + g'_k \\ a_2 = \frac{3}{10}g_k'' \end{array}\right\} \quad (2.6)$$

Substituting (2.6) into (2.2) to obtain

$$\frac{1}{2}g_k''x_k^2 - g_k'x_k + g_k - xg_k''x_k + \frac{1}{2}g_k''x^2 + xg_k' = 0 \quad (2.7)$$

Simplifying (2.7) further, we have,

$$(-x_k + x)g_k' + \frac{1}{2}g_k''x_k^2 + g_k - xg_k''x_k + \frac{1}{2}g_k''x^2 = 0,$$

which implies,

$$x_{k+1} - x_k = -\frac{1}{g_k'}\left(\frac{1}{2}g_k''x_k^2 + g_k - x_{k+1}g_k''x_k + \frac{1}{2}g_k''x_{k+1}^2\right) \quad (2.8)$$

By Newton method

$$x_{k+1} = x_k - \frac{g_k}{g_k'} \quad (2.9)$$

Using (2.9) on (2.8), we obtain,

$$x_{k+1} = x_k - \frac{g_k\left(g_k g_k'' + 2g_k'^2\right)}{g_k'^3}, \quad (2.10)$$

which is the Mamadu Δ^2 iterative scheme.

Let $g(\beta) = 0$. According to Mamadu's Δ^2 iterative scheme, we write $x_k = x_{k-1} - \frac{g_{k-1}\left(g_{k-1}g_{k-1}'' + 2g_{k-1}'^2\right)}{g_{k-1}'^3}$. If $g'(\beta) = g''(\beta) = {}'''g(\beta) \neq 0$, then by Taylor's theorem for some $x_k \leq \sigma \leq \beta$, we have,

$$0 = g(\beta)$$

$$= g(x_k) + (\beta - x_k)g'(x_k) + \frac{(\beta - x_k)^2}{2!}g''(x_k) + \frac{(\beta - x_k)^3}{3!}g'''(x_k) \quad (2.11)$$

$$-g(x_k) = (\beta - x_k)g'(\sigma) + \frac{(\beta - x_k)^2}{2!}g''(\sigma) + \frac{(\beta - x_k)^3}{3!}g'''(\sigma) \quad (2.12)$$

By error definition, we have,

$$e_{k+1} = |\beta - x_{k+1}|$$

$$= \left| \beta - \left(x_k - \frac{g_k\left(g_k g_k^{''} + 2 g_k^{'\,2}\right)}{g_k^{'\,3}} \right) \right|$$

$$= \left| \beta - \left(x_k + \frac{\left((\beta - x_k) g^{'}(\sigma) + \frac{(\beta - x_k)^2}{2!} g^{''}(\sigma) + \frac{(\beta - x_k)^3}{3!} g^{'''}(\sigma)\right)\left(-\left((\beta - x_k) g^{'}(\sigma) + \frac{(\beta - x_k)^2}{2!} g^{''}(\sigma) + \frac{(\beta - x_k)^3}{3!} g^{'''}(\sigma)\right) g^{''}(x_k) + 2 g^{'}(x_k)^2\right)}{g^{'}(x_k)^3} \right) \right| \qquad (2.13)$$

Further simplification and analysis of (2.13) yields

$$= \left| -\frac{1}{36} \frac{g^{'}(\sigma) g^{'''}(\sigma)^2}{g^{'}(x_k)^3} (\beta - x_k)^3 \right|$$

$$= \left| \frac{1}{36} \frac{g^{'}(\sigma) g^{'''}(\sigma)^2}{g^{'}(x_k)^3} \right| |\beta - x_k|^3$$

$$= \left| \frac{1}{36} \frac{g^{'}(\sigma) g^{'''}(\sigma)^2}{g^{'}(x_k)^3} \right| e_n^3$$

In other words, we have

$$\frac{e_{n+1}}{e_n^3} = \left| \frac{1}{36} \frac{g^{'}(\sigma) g^{'''}(\sigma)^2}{g^{'}(x_k)^3} \right|$$

Thus,

$$\lim_{n \to \infty} \left(\frac{e_{n+1}}{e_n^3} \right) = \left| \frac{1}{36} \frac{g^{'}(\sigma) g^{'''}(\sigma)^2}{g^{'}(x_k)^3} \right| = k \neq 0.$$

Hence, the order of convergence of Mamadu's Δ^2 scheme is cubic and the error term (which is asymptotic) is $\left| \frac{1}{36} \frac{g^{'}(\sigma) g^{'''}(\sigma)^2}{g^{'}(x_k)^3} \right| \neq 0$, and $g^{'}(\beta) = g^{''}(\beta) = g'''(\beta) \neq 0$.

Similarly, for $n = 3$ in (2.1) and repeating the above process, we obtain the Mamadu Δ^3 iterative scheme iterative formula as:

$$x_{k+1} = x_k + \frac{g_k\left(g_k^2 g_k^{'''} - 3 g_k g_k^{'} g_k^{''} - 6 g_k^{'\,3}\right)}{6 g_k^{'\,4}}. \qquad (2.14)$$

By using same approach of analysis as above, the rate of convergence of (2.14) is of order 4.

3 Convergence Analysis

We consider the following theorems.

Theorem 3.1. Suppose $g(x) \in [a, b]$ $\forall$ $x \in [a, b]$ defines a differentiable function with the condition $\max|g(x)| = A < 1$, then $x_{k+1} = g(x_k)$ for $k = 0, 1, 2, \ldots$ and $x_0 \in [a, b]$ converges to the required root $\beta \in [a, b]$.

Proof. Let $\beta = \lim_{k \to \infty} x_k$ be given. Then, $\beta = \lim_{k+1 \to \infty} x_k = \lim_{k \to \infty} g(x_k) = g(\lim_{k \to \infty} x_k) = g(\beta)$. Also, for $min\{\beta, x_k\} \leq r \leq max\{\beta, x_k\}$, we have $|\beta - x_{k+1}| = |g(\beta) - g(x_k)| \leq |g^{'}(r)||\beta - x_k| \leq A|\beta - x_k|$. Repeating the process, we obtain $|\beta - x_{k+1}| \leq A^{k+1}|\beta - x_0|$. Since $A < 1$, we have that $\lim_{k+1 \to \infty} A^{k+1} = 0 \Rightarrow$ $\lim_{k+1 \to \infty} |\beta - x_{k+1}| = 0 \Rightarrow \lim_{k \to \infty} x_k = \beta$.

Theorem 3.2. If $g'''(x)$ exist and $x = \beta$ is a root of $g(x) = 0$, then $\exists$ a $\delta > 0$ such that $\{x_k\}_0^\infty$ defined by (2.11) converges to β for any $x_0 \in [\beta - \delta, \beta + \delta]$.

Proof. Obviously $g(\beta) = 0$ and $g''(\beta) = g'''(\beta) \neq 0.$ Since $g'(x)$, $g''(x)$ and $g'''(x)$ are continuous and $g'(x) = g''(x) = g'''(x) \neq 0$, $\exists\ \alpha > 0$ such that $g'(x) = g''(x) = g'''(x) \neq 0\ \forall\ x \in [\beta - \alpha, \beta + \alpha]$. Also, since $g'(\beta) = g''(\beta) = g'''(\beta) = 0$ and $g'(x), g''(x)$ and $g'''(x)$ are continuous, $\exists\ 0 < \delta \leq \alpha$ such that $|g'(x)| < 1$, $|g''(x)| < 1$ and $|g'''(x)| < 1$ for $[\beta - \delta, \beta + \delta]$. Hence, by theorem 3.1, (2.10) and (2.14) converges to β.

4 Numerical Perspectives and Discussion

In this section, we experiment the iterative schemes (2.10) and (2.14) on some algebraic and transcendental equations to show the accuracy, effectiveness and rate of convergence as compared with the Newton-Raphson method.

Example 4.1. Compute the approximate root of $e^x \sin(x) - x^2 = 0$, correct to three decimal places with initial approximation $x_0 = 3$.

Example 4.2. Obtain the root of the $x \log(x) = 1.2$, correct to three decimal places with initial approximation $x_0 = 2$.

Example 4.3: Compute the positive root of the equation $x^2 - 5x + 2 = 0$, correct to four decimal places with $x_0 = 0.5$.

Results for the above examples are presented in the tables below.

Table 1. Computed results for example 4.1

k	**Mamadu Δ^2 method**	**Mamadu Δ^3 method**	**Newton-Raphson method**
0	2.667897090	2.673977394	2.732513710
1	2.618277339	2.618462144	2.631993135
2	2.618039570	2.618039570	2.618254091
3	2.618039570	2.618039570	2.618014029
4	-	-	2.618013958
5	-	-	2.618039570

From Table 1, it requires three iterations for Mamadu Δ^2 and Δ^3 iterative schemes to achieve absolute convergence. Whereas, Newton-Raphson iterative scheme requires six iterations to obtain the required roots. Hence, the computed root is 2.618 correct to 3 decimal places.

Table 2. Computed results for example 4.2

k	**Mamadu Δ^2 method**	**Mamadu Δ^3 method**	**Newton-Raphson method**
0	2.715530523	2.728762614	2.813164836
1	2.740645649	2.740646068	2.741109567
2	2.740646096	2.740646096	2.740646116
3	2.740646096	2.740646096	2.740646097
4	-	-	2.740646096

It was observed that the Newton-Raphson method required five iterations to arrive at the root, whereas Mamadu Δ^2 and Δ^3 iterative methods arrived at the root in three iterations. The computed root of the equation is 2.741 correct to three decimal places.

Table 3. Computed results for example 4.3

k	Mamadu Δ^2 method	Mamadu Δ^3 method	Newton-Raphson method
0	0.43847656250	0.43847656250	0.4375000000
1	0.43834471871	0.43834471871	0.4384469697
2	0.43834471871	0.43834471871	0.4384471873
3	-	-	0.4384471871

Here, the Mamadu Δ^2 and Δ^3 iterative methods required only two iterations to obtain the required root, whereas the Newton-Raphson scheme required four iterations to arrive at the computed root. Thus, the computed root for the equation is 0.4384 correct to 4 decimal places.

5 Conclusion

In this article, we have considered orthogonal based iterative schemes for seeking the approximate roots of algebraic and transcendental equations. The new derived iterative schemes named “Mamadu Δ^2 and Δ^3 iterative schemes” perform incredibly well requiring just few iterations to obtain the required roots as shown in the Tables 1 – 3. The Mamadu Δ^2 and Δ^3 iterative schemes in comparison with the Newton-Raphson method converge faster and better. Thus, it is recommended that these new iterative schemes be incorporated as new iterative methods for finding the roots of nonlinear equations.

Competing Interests

Author has declared that no competing interests exist.

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