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On the Superstability of Trigonometric Type Functional Equations

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Abstract

The aim of this paper is to study the superstability for the mixed trigonometric functional equation:

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \qquad (E_{f,g})$$

and the stability of the Pexider type functional equation:

$$f(xy) - f(x\sigma(y)) = 2g(x)h(y), \quad x, y \in G, \qquad (E_{f,g,h})$$

where G is any group, not necessarily abelian, f, g and h are unknown complex valued functions and σ is an involution of G. As a consequence we prove that if f satisfies the inequality $|f(xy) - f(x\sigma(y)) - 2f(x)f(y)| \le \delta$ for all $x, y \in G$ then f is bounded.

Keywords: *Superstability; d'Alembert's equation; Trigonometric functional equation;2000 Mathematics Subject Classification; Primary 39B72.*

1 Introduction

J. Baker, J. Lawrence and F. Zorzitto in [1] introduced that if f satisfies the stability inequality

$$|E_1(f) - E_2(f)| \le \varepsilon,$$

then either f is bounded or $E_1(f) = E_2(f)$. The stability of this type is called the superstability.

In [2,3,4] D. Zeglami, A. Roukbi and S. Kabbaj proved the superstability of the Wilson's functional equation

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$$f(xy)+f(x\sigma(y))=2f(x)g(y), \quad x,y\in G, \qquad (W)$$

and the d'Alembert's functional equation

$$f(xy)+f(x\sigma(y))=2f(x)f(y), \quad x,y\in G,$$

where G is any group and σ is an involution of G. Namely, the following theorem holds true.

Theorem 1. *Let $\delta>0$ be given. Assume that functions $f,g:G\to C$ satisfy the inequality*

$$\left|f(xy)+f(x\sigma(y))-2f(x)g(y)\right|\le\delta \quad \text{for all } x,y\in G.$$

Then

i) f,g are bounded or

ii) f is unbounded and g satisfies the d'Alembert's long functional equation

$$g(xy)+g(x\sigma(y))+g(yx)+g(\sigma(y)x)=4g(x)g(y) \quad or$$

iii) g is unbounded and the pair (f,g) satisfies the equation (W).

The superstability of the trigonometric functional equation concerned with the sine and the cosine equations

$$f(x+y)-f(x-y)=2f(x)f(y), \quad x,y\in G,$$
$$f(x+y)-f(x-y)=2f(x)g(y), \quad x,y\in G,$$
$$f(x+y)-f(x-y)=2g(x)f(y), \quad x,y\in G,$$
$$f(x+y)-f(x-y)=2g(x)h(y), \quad x,y\in G,$$

where $(G,+)$ is an abelian group, was investigated by Kim [5,6] and Kim and Lee [7].

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, and some exponential functions also satisfy the above mentioned equations, thus they can be called by the hyperbolic cosine sine, trigonometric, exponential functional equations, respectively.

For example,

$$\cosh(x+y)-\cosh(x-y)=2\sinh(x)\sinh(y),$$
$$\sinh(x+y)-\sinh(x-y)=2\cosh(x)\sinh(y),$$
$$\sinh^2(x+y)-\sinh^2(x-y)=\sinh(2x)\sinh(2y),$$
$$e^{x+y}-e^{x-y}=2e^x\sinh(y),$$
$$ca^{x+y}-ca^{x-y}=2ca^x\,\frac{a^y-a^{-y}}{2}.$$

The aim of this paper is to investigate the superstability problem for the mixed trigonometric functional equations

$$f(xy)-f(x\sigma(y))=2f(x)f(y), \quad x,y\in G, \qquad (T)$$
$$f(xy)-f(x\sigma(y))=2f(x)g(y), \quad x,y\in G, \qquad (T_{f,g})$$

$$f(xy) - f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \qquad (T_{g,f})$$

$$f(xy) - f(x\sigma(y)) = 2g(x)h(y), \quad x, y \in G, \qquad (T_{f,g,h})$$

where G is any group, not necessarily abelian, the unknown functions f, g, h are to be determined and σ is an involution of G, i. e. $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$. The interested reader should refer to $[1-20]$ for a thorough account on the subject of stability of functional equations and to $[21]$ for solutions of the functional equation $(T_{f,g,h})$ in the case that G is an abelian group.

In this paper, let G be any group, σ is an involution of G, C the field of complex numbers and δ is a nonnegative real constant. We may assume that f and g are complex valued functions on G and we denote by $\breve{f}$ the function defined by $\breve{f}(x) := f(x^{-1})$, for all $x \in G$.

2. Superstability of the Equation $(T_{f,g})$

We start with solutions of the functional equation (T).

Lemma 1. The solution of the functional equation

$$f(xy) - f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \qquad (T)$$

on any group G is the zero function $f \equiv 0$.

Proof. Putting $y = e$ in (T) we get $f(e) = 0$. Setting $x = e$ in (T) we have $f(\sigma(y)) = f(y)$ for all $y \in G$. From (T) and the equality

$$f(x\sigma(y)) - f(xy) = 2f(x)f(\sigma(y)), \quad x, y \in G,$$

we obtain that

$$f(x)(f(\sigma(y) + f(y)) = 0, \quad \textit{for all } x, y \in G,$$

from which we conclude that $f(\sigma(y)) = -f(y)$ for all $y \in G$. Consequently we have $f(\sigma(y)) = -f(y) = f(y)$ i.e. $f \equiv 0$ is the only solution of (T).

Lemma 2. *Let $\delta > 0$ be given. Assume that functions $f, g : G \to C$ satisfy the inequality*

$$\left| f(xy) - f(x\sigma(y)) - 2f(x)g(y) \right| \le \delta, \textit{ for all } x, y \in G \qquad (2.1)$$

such that $f \neq 0$. If g is unbounded then so is f.

Proof. Assume that g is unbounded function satisfying the inequality (2.1). If $f \neq 0$ is bounded, let $M = \sup|f|$ and choose $a \in G$ such that $f(a) \neq 0$ then we get from the inequality (2.1) that $|g(x)| \leq \frac{1}{2f(a)}(2M + \delta)$ for all $x \in G$, i.e. g is bounded too which contradicts our assumption.

In Theorem 2, the superstability of the equation $(T_{f,g})$ will be investigated.

Theorem 2. *Let* $\delta > 0$ *be given. Assume that functions* $f, g : G \to C$ *satisfy the inequality*

$$\left| f(xy) - f(x\sigma(y)) - 2f(x)g(y) \right| \leq \delta, \tag{2.2}$$

for all $x, y \in G$ *. Then*

i) f, g *are bounded or*

ii) f *is unbounded and* g *satisfies the functional equation*

$$g(xy) - g(x\sigma(y)) - g(yx) + g(\sigma(y)x) = 0, \tag{2.3}$$

or

iii) g is unbounded and the pair (f, g) *satisfies the equation* $(T_{f,g})$. *if* $f \neq 0$, *then* g *satisfies the equation (2.3).*

Proof. *Assume that* f, g *satisfy inequality (2.2). First we consider the case of* f *unbounded. For all* $x, y, z \in G$ *we have*

$$\begin{aligned}
2|f(z)||g(xy) - g(x\sigma(y)) - g(yx) + g(\sigma(y)x)| \\
&= |2f(z)g(xy) + 2f(z)g(x\sigma(y)) - 2f(z)g(yx) - 2f(z)g(\sigma(y)x)| \\
&\leq |-f(zxy) + f(z\sigma(y)\sigma(x)) + 2f(z)g(xy)| \\
&+ |f(zx\sigma(y)) - f(zy\sigma(x)) - 2f(z)g(x\sigma(y))| \\
&+ |f(zyx) - f(z\sigma(x)\sigma(y)) - 2f(z)g(yx)| \\
&+ |-f(z\sigma(y)x) + f(z\sigma(x)y) + 2f(z)g(\sigma(y)x)| \\
&+ |f(zxy) - f(zx\sigma(y)) - 2f(zx)g(y)| \\
&+ |-f(zyx) + f(zy\sigma(x)) + 2f(zy)g(x)| \\
&+ |f(z\sigma(y)x) - f(z\sigma(y)\sigma(x)) - 2f(z\sigma(y))g(x)| \\
&+ |-f(z\sigma(x)y) + f(z\sigma(x)\sigma(y)) + 2f(z\sigma(x))g(y)| \\
&+ 2|g(y)||f(zx) - f(z\sigma(x)) - 2f(z)g(x)| \\
&+ 2|g(x)||-f(zy) + f(z\sigma(y)) + 2f(z)g(y)|.
\end{aligned}$$

By virtue of inequality (2.2), we have

$$2|f(z)||g(xy)-g(x\sigma(y))-g(yx)+g(\sigma(y)x)| \le 8\delta + 2(|g(x)|+|g(y)|)\delta. \qquad (2.4)$$

If we fix x, y, the right hand side of the above inequality is bounded function of z. Since f is unbounded, from (2.4), we conclude that

$$g(xy)-g(x\sigma(y))-g(yx)+g(\sigma(y)x)=0 ,$$

which ends the proof in this case.

If g is unbounded, then for $f=0$ the pair (f,g) is a trivial solution of the equation $(T_{f,g})$.

Now assume that $f \ne 0$. For all $x, y, z \in G$ we have

$$\begin{aligned}
2|g(z)||f(xy)-f(x\sigma(y))-2f(x)g(y)| \\
&= |2g(z)f(xy)-2g(z)f(x\sigma(y))-4g(z)f(x)g(y)| \\
&\le |-f(xyz)+f(xy\sigma(z))+2f(xy)g(z)| \\
&+|f(x\sigma(y)z)-f(x\sigma(y)\sigma(z))-2f(x\sigma(y))g(z)| \\
&+|f(xyz)-f(x\sigma(z)\sigma(y))-2f(x)g(yz)| \\
&+|-f(xy\sigma(z))+f(xz\sigma(y))+2f(x)g(y\sigma(z))| \\
&+|f(x\sigma(z)y)-f(x\sigma(y)z)-2f(x)g(\sigma(z)y)| \\
&+|-f(xzy)+f(x\sigma(y)\sigma(z))+2f(x)g(zy)| \\
&+|-f(x\sigma(z)y)+f(x\sigma(z)\sigma(y))+2f(x\sigma(z))g(y)| \\
&+|f(xzy)-f(xz\sigma(y))-2f(xz)g(y)| \\
&+|2f(x)\{g(yz)-g(y\sigma(z))-g(zy)+g(\sigma(z)y)\}| \\
&+2|g(y)||f(xz)-f(x\sigma(z))-2f(x)g(z)|.
\end{aligned}$$

In virtue of inequalities (2.2) we obtain

$$\begin{aligned}
2|g(z)||f(xy)-f(x\sigma(y))-2f(x)g(y)| \\
&\le 8\delta + 2\delta|g(y)| + 2|f(x)||g(yz)-g(y\sigma(z))-g(zy)+g(\sigma(z)y)|.
\end{aligned}$$

By using Lemma 2 we see that g is unbounded implies necessarily that f is unbounded hence according to theorem 2 ii) g is a solution of the equation (2.3). So we conclude that

$$2|g(z)||f(xy)-f(x\sigma(y))-2f(x)g(y)| \le 8\delta + 2\delta|g(y)|. \qquad (2.5)$$

Again the right hand side of (2.5) as a function of z is bounded for all fixed x, y. Since g is unbounded, from (2.5), we see that the pair (f,g) satisfies the equation $(T_{f,g})$ and it is easy to get that if $f \ne 0$, then g satisfies (2.3) which finished the proof of the theorem 2.

As an immediate consequence of Theorem 2, we have the following result which has been the subject of [7] in the case where G is an abelian group.

Corollary 1. *Let $\delta > 0$ be given. Assume that the function $f : G \to C$ satisfies the inequality*

$$\left| f(xy) - f(x\sigma(y)) - 2f(x)f(y) \right| \le \delta, \tag{2.6}$$

for all $x, y \in G$. Then f is bounded.

Proof. Define $f = g$ in the case (iii) of Theorem 2 we get that either f is bounded or

$$f(xy) - f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G.$$

The rest of the proof follows from Lemma 1.

3. Application: Stability of the Equation $(T_{f,g,h})$

Lemma 3. *Let $\delta > 0$ be given. Assume that functions f, g and $h : G \to C$ satisfy the inequality*

$$\left| f(xy) - f(x\sigma(y)) - 2g(x)h(y) \right| \le \delta, \tag{3.1}$$

for all $x, y \in G$. Then

i) If g is unbounded then $\breve{h} = -h$.

ii) If $h(e) = 1$ then

$$\left| g(xy) - g(x\sigma(y)) - 2g(x)\tilde{h}(y) \right| \le \delta \text{ and } \left| g(xy) + g(x\sigma(y)) - 2g(x)\tilde{h}(y) \right| \le \delta$$

where $\tilde{h}(x) = \dfrac{h(x) + h(\sigma(x))}{2}$, $x \in G$.

Proof. Assume that g is an unbounded function satisfying (3.1). From the inequalities

$$\left| f(xy) - f(x\sigma(y)) - 2g(x)h(y) \right| \le \delta$$

and

$$\left| f(x\sigma(y)) - f(xy) - 2g(x)h(\sigma(y)) \right| \le \delta$$

we get that

$$\left| 2g(x) \right| \left| h(\sigma(y)) + h(y) \right| \le 2\delta$$

for all $x, y \in G$. Hence we obtain that $\breve{h}(y) = -h(y)$ for all $y \in G$ because g is unbounded.

(ii) Assume that $h(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show that

$$\left| g(x) \right| \le \frac{\delta}{2}, \quad x \in G, \tag{3.2}$$

i.e. g is bounded. For all $x, y \in G$ we have

$$\left|g(xy)-g(x\sigma(y))-2g(x)\frac{h(y)+h(\sigma(y))}{2}\right|=\left|g(xy)-g(x\sigma(y))-g(x)h(y)-g(x)h(\sigma(y))\right|$$
$$\leq\frac{1}{2}\left|f(xy)-f(x\sigma(y)-2g(x)h(y)\right|$$
$$+\frac{1}{2}\left|f(x\sigma(y))-f(xy)-2g(x)h(\sigma(y))\right|$$
$$+\left|g(xy)\right|+\left|g(x\sigma(y)\right|.$$

In virtue of inequalities (3.1) and (3.2), we obtain

$$\left|g(xy)-g(x\sigma(y))-2g(x)\frac{h(y)+h(\sigma(y))}{2}\right|\leq\frac{\delta}{2}+\frac{\delta}{2}+\frac{\delta}{2}+\frac{\delta}{2}=2\delta.$$

And similarly we have

$$\left|g(xy)+g(x\sigma(y))-2g(x)\frac{h(y)+h(\sigma(y))}{2}\right|\leq\frac{1}{2}\left|f(xy)-f(x\sigma(y)-2g(x)h(y)\right|$$
$$+\frac{1}{2}\left|f(x\sigma(y))-f(xy)-2g(x)h(\sigma(y))\right|$$
$$+\left|g(xy)\right|+\left|g(x\sigma(y)\right|$$
$$\leq 2\delta,$$

for all $x,y\in G$.

In Theorem 3, the stability of the equation $(T_{f,g,h})$, under the condition $h(e)\neq 0$, will be investigated on an arbitrary group.

Theorem 3. *Let $\delta>0$ be given. Assume that functions f,g and $h:G\to C$ with $h(e)\neq 0$ satisfy the inequality*

$$\left|f(xy)-f(x\sigma(y))-2g(x)h(y)\right|\leq\delta,$$

for all $x,y\in G$. Then either the function $\frac{h+\breve{h}}{2}$ is bounded or the pair (g,h) satisfies the equation

$$g(xy)=g(x)\frac{h(y)+h(\sigma(y))}{h(e)}, \tag{3.3}$$

for all $x,y\in G$. Consequently, if $g\neq 0$ then the functions g and $\frac{h+\breve{h}}{2}$ are bounded.

Proof. Assume that f,g and h satisfy the inequality (3.1) with $h(e)\neq 0$ then g is bounded. Dividing the two sides of the inequality (3.1) by $\alpha=h(e)$ we find that

$$\left|\tilde{f}(xy)-\tilde{f}(x\sigma(y))-2g(x)\tilde{h}(y)\right|\leq\frac{\delta}{|\alpha|},\ for\ all\ x,y\in G,$$

where $\tilde{f}=\dfrac{f}{\alpha}$ and $\tilde{h}=\dfrac{h}{\alpha}$. We see that $\tilde{h}(e)=1$. By using Lemma 3 (ii) we obtain that

$$\left|g(xy)-g(x\sigma(y))-2g(x)\frac{\tilde{h}(y)+\tilde{h}(\sigma(y))}{2}\right|\leq 2\frac{\delta}{|\alpha|},\ x,y\in G$$

and

$$\left|g(xy)+g(x\sigma(y))-2g(x)\frac{\tilde{h}(y)+\tilde{h}(\sigma(y))}{2}\right|\leq 2\frac{\delta}{|\alpha|},\ x,y\in G.$$

Using, respectively Theorem 2 and Theorem 1, we conclude that if $\dfrac{h+\breve{h}}{2}$ is unbounded then the pair (g,h) satisfies the equations

$$g(xy)-g(x\sigma(y))=g(x)\frac{h(y)+h(\sigma(y))}{h(e)} \tag{3.4}$$

and

$$g(xy)+g(x\sigma(y))=g(x)\frac{h(y)+h(\sigma(y))}{h(e)} \tag{3.5}$$

by adding (3.4) and (3.5) we get that the pair (g,h) satisfies (3.3).
Now, assume that $g\neq 0$. Putting $y=e$ in the inequality (3.1). It is easy to show that

$$|g(x)h(e)|\leq\frac{\delta}{2},\quad x\in G,$$

i.e. g is bounded because $h(e)\neq 0$. The equality (3.3) implies that the function $\dfrac{h+\breve{h}}{2}$ is also bounded.

The following corollary is a particular case of Theorem 3.

Corollary 2. *Let* $\delta>0$ *be given. Assume that functions* $f,g:G\to C$ *with* $f(e)\neq 0$ *satisfy the inequality*

$$|f(xy)-f(x\sigma(y))-2g(x)f(y)|\leq\delta,$$

for all $x,y\in G$. *Then either the function* $\tilde{f}=\dfrac{f+\breve{f}}{2}$ *is bounded or the pair* (f,g) *satisfies the equation*

$$g(xy)=g(x)\frac{f(y)+f(\sigma(y))}{f(e)},$$

for all $x, y \in G$. Consequently, if $g \neq 0$ then the functions g and $\frac{f+\breve{f}}{2}$ are bounded.

4. Remark

The results of this paper also can be extended to the stability of the considered equations controlled even by variable bounds.

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Competing Interests

Authors have declared that no competing interests exist.

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