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Hypergeometric Representation of Modified Beta Stancu Operators

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Abstract

In the present paper, we extend the results of [1] by applying hypergeometric series. We study local and global properties, Voronovskaja type asymptotic formula and error estimation for modified Beta operators. We also derive some other characteristics of these operators.

Keywords: Hypergeometric series, Voronovskaja type asymptotic formula, Modified-Beta Stancu operators, Estimation of error, Direct theorem.

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1 Introduction

To approximate integrable function on $[0, \infty)$, the modified Beta operators given by Gupta and Ahmad [1] are defined as

$$B_n(f, x) = \left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) dt, \qquad x \in [0, \infty) \tag{1.1}$$

where

$$b_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!} \frac{x^k}{(1+x)^{n+k+1}}$$

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and

$$p_{n,k}(t)=\frac{(n+k-1)!}{k!(n-1)!}\frac{t^k}{(1+t)^{n+k}}.$$

It was observed in [1], that by taking the modification of Beta operators defined in (1.1), we can have better approximation.
Some approximation properties for these operators and other similar type of operators were studied in [2-12].
Now [13] studied certain Beta type operators in the hypergeometric form and such notations can be studied in [14]. Some important aspects of approximation theory can be studied in the book written by [15].
Using Pochhammer symbol $(n)_k$ defined as $(n)_k = n(n+1)(n+2)(n+3)..(n+k-1)$, we can write operators (1.1) as

$$\begin{aligned} B_n(f,x) &= \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}\frac{(n)(n+1)_k}{k!}\frac{x^k(1+x)^{-1}}{(1+x)^{n+k}}\int_0^\infty\frac{(n)_k}{k!}\frac{t^k}{(1+t)^{n+k}}f(t)dt \\ &= (n-1)\int_0^\infty\frac{f(t)(1+x)^{-1}}{[(1+x)(1+t)]^n}\sum_{k=0}^{\infty}\frac{(n)_k(n+1)_k(xt)^k}{(k!)^2[(1+x)(1+t)]^k}dt. \end{aligned}$$

Using the hypergeometric series ${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$ and the equality $(1)_k = k!$, we have

$$B_n(f,x)=(n-1)\int_0^\infty\frac{f(t)(1+x)^{-1}}{[(1+x)(1+t)]^n}{}_2F_1\left(n,n+1;1;\frac{xt}{(1+x)(1+t)}\right)dt.$$

Now using ${}_2F_1(a,b;c;x)$ = ${}_2F_1(b,a;c;x)$ and applying Pfaff-Kummer transformation,

$${}_2F_1(a,b;c;x)=(1-x)^{-a}{}_2F_1(a,c-b;c;\frac{x}{x-1}),$$

we have

$$B_n(f,x)=(n-1)\int_0^\infty\frac{f(t)(1+x)^{-1}}{(1+x+t)^n}{}_2F_1(n,-n;1;\frac{-xt}{1+x+t})dt. \tag{1.2}$$

which is another from of the operator (1.1) in terms of hypergeometric functions. The Stancu type generalization of Bernstein operators was given in [16].
For $0\le\alpha\le\beta$, we propose the Stancu type generalization of modified-Beta operators,

$$B_{n,\alpha,\beta}(f,x)=(n-1)\int_0^\infty f\left(\frac{nt+\alpha}{n+\beta}\right)\frac{(1+x)^{-1}}{(1+x+t)^n}{}_2F_1\left(n,-n;1;\frac{-xt}{1+x+t}\right)dt. \tag{1.3}$$

As a special case if $\alpha=\beta=0$, the operators (1.3) reduce to modified Beta operators defined in (1.1).
Let us consider

$$C_\gamma[0,\infty)=\{f\in C[0,\infty): f(t)=O(t^\gamma),\gamma>0\}.$$

The operators $B_{n,\alpha,\beta}(f,x)$ are well defined for $f\in C_\gamma[0,\infty)$.
Now [17] studied rate of approximation and weighted approximation of modified Beta operators. [18] introduced some direct results in simultaneous approximation for Baskakov-Durrmeyer-Stancu operators and [19] obtained a new type of Baskakov- Durrmeyer operators by taking the weight function of Beta operators. In the present paper, we establish moments of modified Beta Stancu operators using the technique of hypergeometric series. Next, we study Voronovkaja type asymptotic formula and an estimation of error in simultaneous approximation for the MBS operators.

2 Moment Estimation and Auxiliary Results

In this section, we establish certain lemmas which will be useful for the proof of our main theorems.

Lemma 1. For $n \in N$ and $r > -1$, we have

$$B_n(t^r, x) = (n-1)\frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n)}(1+x)^r {}_2F_1\left(-n, -r; 1; \frac{x}{1+x}\right). \tag{2.1}$$

Moreover

$$B_n(t^r, x) = \frac{(n-r-2)!(n+r)!}{n!(n-2)!}x^r + r^2\frac{(n-r-2)!(n+r-1)!}{n!(n-2)!}x^{r-1} + O(n^{-2}) \tag{2.2}$$

Proof: Taking $f(t) = t^r$ and $t = (1+x)u$, by using Pfaff-Kummer transformation in (1.2), we have

$$\begin{aligned}
B_n(t^r, x) &= (n-1)\int_0^\infty \frac{(1+x)^r u^r}{[(1+x)(1+u)]^n}\sum_{k=0}^\infty \frac{(n)_k(-n)_k}{(k!)^2}\frac{(-x(1+x)u)^k}{[(1+x)(1+u)]^k}du \\
&= (n-1)\sum_{k=0}^\infty \frac{(n)_k(-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n}\int_0^\infty \frac{u^{r+k}}{(1+u)^{n+k}}du \\
&= (n-1)\sum_{k=0}^\infty \frac{(n)_k(-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n}B(r+k+1, n-r-1) \\
&= (n-1)\sum_{k=0}^\infty \frac{(n)_k(-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n}\frac{\Gamma(r+k+1)\Gamma(n-r-1)}{\Gamma(n+k)}.
\end{aligned}$$

Again taking $\Gamma(n+k+1) = \Gamma(n+1)(n+1)_k$, we get

$$\begin{aligned}
B_n(t^r, x) &= (n-1)\sum_{k=0}^\infty \frac{(n)_k(-n)_k}{(k!)^2}(-x)^k(1+x)^{r-n}\frac{\Gamma(r+1)(r+1)_k\Gamma(n-r-1)}{\Gamma(n)(n)_k} \\
&= (n-1)(1+x)^{r-n}\frac{\Gamma(r+1)\Gamma(n-r-1)}{\Gamma(n)}\sum_{k=0}^\infty \frac{(-n)_k(r+1)_k}{(k!)^2}(-x)^k \\
&= (n-1)(1+x)^{r-n}\frac{\Gamma(r+1)\Gamma(n-r-1)}{\Gamma(n)}{}_2F_1(-n, r+1; 1; -x).
\end{aligned}$$

Using Pfaff- Kummer transformation, we get

$$B_n(t^r, x) = (n-1)(1+x)^r\frac{\Gamma(r+1)\Gamma(n-r-1)}{\Gamma(n)}{}_2F_1\left(-n, -r; 1; \frac{x}{x-1}\right).$$

$$\begin{aligned}
B_n(t^r, x) &= (n-1)(1+x)^r\frac{r!(n-r-2)!}{(n-1)!}{}_2F_1\left(-r, 1+n; 1; \frac{x}{x-1}\right). \\
&= \frac{r!(n-r-2)!}{(n-2!}\sum_{k=o}^\infty \frac{(-r)_k(n+1)_k}{(k!)^2}(-x)^k \\
&= \frac{(r!)^2(n-r-2)!}{n!(n-2!}\sum_{k=o}^\infty \frac{(n+k)!}{(r-k)!(k!)^2}(x)^k \\
B_n(t^r, x) &= \frac{(n-r-2)!(n+r)!}{n!(n-2)!}x^r + r^2\frac{(n-r-2)!(n+r-1)!}{n!(n-2)!}x^{r-1} + O(n^{-2})
\end{aligned}$$

Lemma 2. For $0 \leq \alpha \leq \beta$, we have

$$\begin{aligned}
&B_{n,\alpha,\beta}(t^r,x) = \\
&x^r \frac{n^r}{(n+\beta)^r} \frac{(n+r)!(n-r-2)!}{n!(n-2)!} \\
&+x^{r-1}\left\{r^2 \frac{n^r}{(n+\beta)^r} \frac{(n+r-1)!(n-r-2)!}{n!(n-2)!} + r\alpha \frac{n^{r-1}}{(n+\beta)^r} \frac{(n+r-1)!(n-r-1)!}{n!(n-2)!}\right\} \\
&+x^{r-2}\left\{r(r-1)^2\alpha \frac{n^{r-1}}{(n+\beta)^r} \frac{(n+r-2)!(n-r-1)!}{n!(n-2)!} + \frac{r(r-1)\alpha^2}{2} \frac{n^{r-2}}{(n+\beta)^r} \frac{(n+r-2)!(n-r)!}{n!(n-2)!}\right\} \\
&+O(n^{-2}).
\end{aligned}$$

Proof: By using Binomial theorem, the relation between operators (1.2) and (1.3) can be defined as

$$\begin{aligned}
B_{n,\alpha,\beta}(t^r,x) &= (n-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^r dt \\
&= (n-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}\frac{(nt)^j\alpha^{r-j}}{(n+\beta)^r} dt \\
&= \sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}\frac{n^j\alpha^{r-j}}{(n+\beta)^r} B_n(t^j,x).
\end{aligned}$$

Applying this on (2.2), we get the required result.

Lemma 3. [1] For $m \in N\bigcup\{0\}$, if

$$T_{n,m}(x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m.$$

then there exists the following recurrence relation

$$(n+1)T_{n,m+1}(x) = x(1+x)[T'_{n,m}(x) + mT_{n,m-1}(x)].$$

Consequently,

1. $T_{n,m}(x)$ are polynomials in x of degree at most m.
2. $T_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integral part of α.

Lemma 4. We define the central moments as

$$T_{n,m}(x) = B_{n,\alpha,\beta}((t-x)^m,x) = \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt, \qquad m \in N\{0\},$$

We have the following recurrence relation, for $n > m+1$.

$$\begin{aligned}
(n-m-2)\left(\frac{n+\beta}{n}\right)T_{n,m+1}(x) &= x(1+x)[T'_{n,m}(x) + mT_{n,m-1}(x)] \\
&+\left[(m+nx+1+x) + \left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta} - x\right)(n-2m-2)\right]T_{n,m}(x) \\
&-\left(\frac{\alpha}{n+\beta} - x\right)\left[1 - \left(\frac{\alpha}{n+\beta} - x\right)\left(\frac{n+\beta}{n}\right)\right]mT_{n,m-1}(x).
\end{aligned}$$

We can obtain first three moments as $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{x[3n+\beta(2-n)+n+\alpha(n-2)]}{(n+\beta)(n-2)}$,

$$\begin{aligned} T_{n,2}(x) &= \frac{x^2[2n^2(n+7) - n^2\beta(7-n) + n\beta(n\beta - 5\beta + 18 + 6\beta^2]}{(n+\beta)^2(n-2)(n-3)} \\ &+ \frac{x\left[2n^2(n+5) + 2n^2\alpha(5-2n) - 2n^2\beta(1+\alpha) + 2n(5\alpha\beta + 3\beta - 9\alpha) - 12\alpha\beta\right]}{(n+\beta)^2(n-2)(n-3)} \\ &+ \frac{2n^2 + n^2\alpha(\alpha+2) - n\alpha(5\alpha+6) + 6\alpha^2}{(n+\beta)^2(n-2)(n-3)}. \end{aligned}$$

From the recurrence relation, it is easily verified that for all $x \in [0, \infty)$, we have

$$T_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Proof: $T_{n,0}(x) = 1$ (by the definition of the operators (1.3)). For the proof of other moments we follow the recurrence relation. Now we prove the recurrence relation as follows.
Using the identities

$$x(1+x)b'_{n,k}(x) = [k - (n+1)x]b_{n,k}(x)$$

$$t(1+t)p'_{n,k}(t) = (k - nt)p_{n,k}(t),$$

We have

$$x(1+x)T'_{n,m}(x) = \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}[k-(n+1)x]b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt - mx(1+x)T_{n,m-1}(x).$$

Thus

$$\begin{aligned} & x(1+x)[T'_{n,m}(x) + mT_{n,m-1}(x)] \\ = & \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}(k-nx)b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt \\ & -\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}xb_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt \\ = & \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}[(k-nt) + n(t-x)]p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt - xT_{n,m}(x) \\ = & \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty} t(1+t)p'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt \\ & +\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty} n(t-x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt - xT_{n,m}(x). \end{aligned}$$

Now writing

$$t = \left(\frac{n+\beta}{n}\right)\left[\frac{nt+\alpha}{n+\beta} - x - \left(\frac{\alpha}{n+\beta} - x\right)\right],$$

We have

$$\begin{aligned}
& x(1+x)[T'_{n,m}(x)+mT_{n,m-1}(x)] \\
= & \left(\frac{n+\beta}{n}\right)\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt \\
& -\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m}dt \\
& +\left(\frac{n+\beta}{n}\right)^2[\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+2}dt \\
& +\left(\frac{\alpha}{n+\beta}-x\right)^2\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m}dt \\
& -2\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt] \\
& +n\left(\frac{n+\beta}{n}\right)T_{n,m+1}(x)-n\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)T_{n,m}(x) \\
& -nxT_{n,m}(x)-xT_{n,m}(x).
\end{aligned}$$

Integrating by parts and by simple computation, we get

$$\begin{aligned}
(n-m-2)\frac{n+\beta}{n}T_{n,m+1}(x) &= x(1+x)[T'_{n,m}(x)+mT_{n,m-1}(x)] \\
&\quad +\left[(m+nx+1+x)+\frac{n+\beta}{n}\left(\frac{\alpha}{n+\beta}-x\right)(n-2m-2)\right]T_{n,m}(x) \\
&\quad -\left(\frac{\alpha}{n+\beta}-x\right)\left[1-\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)\right]mT_{n,m-1}(x).
\end{aligned}$$

Lemma 5. [11] There exist the polynomials $q_{i,j,r}(x)$ on $[0,\infty)$, independent of n and k such that

$$x(1+x))^r\frac{d^r}{dx^r}b_{n,k}(x)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+1)^i(k-(n+1)x)^j q_{i,j,r}(x)b_{n,k}(x).$$

3 Direct Estimates

In this section, we propose some direct results including asymptotic formula and an error estimation in simultaneous approximation.

Theorem 1. Let $f\in C_\gamma[0,\infty)$ be bounded on every finite sub-interval $[0,\infty)$ admitting the derivative of order $(r+2)$ at a fixed $x\in[0,\infty)$. Let $f(t)=O(t^\gamma)$ as $t\to\infty$ for some $\gamma>0$, then we have

$$\lim_{n\to\infty} n\left(B^{(r)}_{n,\alpha,\beta}(f,x)-f^{(r)}(x)\right)=r(r+2-\beta)f^{(r)}(x)+[(1+r+\alpha)+x(2r+3-\beta)]f^{(r+1)}(x)+x(1+x)f^{(r+2)}(x),$$

Proof: By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and $\varepsilon(t,x)=0((t-x)^\delta)$ as $t\to\infty$ for some $\delta>0$, therefore we can write,

$$n[B^{(r)}_{n,\alpha,\beta}(f,x)-f^{(r)}(x)]=n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}B^{(r)}_{n,\alpha,\beta}((t-x)^i,x)-f^{(r)}(x)\right]+nB^{(r)}_{n,\alpha,\beta}(\varepsilon(t,x)(t-x)^{r+2},x)=:E_1+E_2$$

From Lemma 2, we have

$$\begin{aligned}E_1 &= n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}B_{n,\alpha,\beta}^{(r)}(t^j,x)-nf^{(r)}(x)=\frac{f^{(r)}(x)}{r!}n\left(B_{n,\alpha,\beta}^{(r)}(t^r,x)-r!\right)\\&+\frac{f^{(r+1)}(x)}{(r+1)!}n\left\{(r+1)(-x)B_{n,\alpha,\beta}^{(r)}(t^r,x)+B_{n,\alpha,\beta}^{(r)}(t^{r+1},x)\right\}\\&+\frac{f^{(r+2)}(x)}{(r+2)!}n\left\{\frac{(r+2)(r+1)}{2}x^2B_{n,\alpha,\beta}^{(r)}(t^r,x)+(r+2)(-x)B_{n,\alpha,\beta}^{(r)}(t^{r+1},x)+B_{n,\alpha,\beta}^{(r)}(t^{r+2},x)\right\}\end{aligned}$$

$$\begin{aligned}&= n\left[\frac{n^r(n+r)!(n-r-2)!}{(n+\beta)^r n!(n-2)!}-1\right]f^{(r)}(x)+n\frac{f^{(r+1)}(x)}{(r+1)!}\{(r+1)(-x)\frac{n^r(n+r)!(n-r-2)!}{(n+\beta)^r n!(n-2)!}r!\\&+\frac{n^{r+1}(n+r+1)!(n-r-3)!}{(n+\beta)^{r+1}n!(n-2)!}(r+1)!x+\frac{(r+1)^2n^{r+1}(n+r)!(n-r-3)!}{(n+\beta)^{r+1}n!(n-2)!}r!\\&+(r+1)\alpha\frac{n^r(n+r)!(n-r-2)!}{(n+\beta)^{r+1}n!(n-2)!}r!\}+n\frac{f^{(r+2)}(x)}{(r+2)!}\{\frac{(r+2)(r+1)}{2}x^2\frac{n^r(n+r)!(n-r-2)!}{(n+\beta)^r n!(n-2)!}r!\\&-(r+2)x(\frac{n^{r+1}(n+r+1)!(n-r-3)!}{(n+\beta)^{r+1}n!(n-2)!}(r+1)!x+\frac{(r+1)^2n^{r+1}(n+r)!(n-r-3)!}{(n+\beta)^{r+1}n!(n-2)!}r!\\&+(r+1)\alpha\frac{n^r(n+r)!(n-r-3)!}{(n+\beta)^{r+1}n!(n-2)!}r!)+\frac{n^{r+2}(n+r+2)!(n-r-4)!}{(n+\beta)^{r+2}n!(n-2)!}\frac{(r+2)!}{2}x^2\\&+\frac{(r+2)^2n^{r+2}(n+r+1)!(n-r-4)!}{(n+\beta)^{r+2}n!(n-2)!}(r+1)!x+(r+2)\alpha\frac{n^{r+1}(n+r+1)!(n-r-3)!}{(n+\beta)^{r+2}n!(n-2)!}(r+1)!x\\&+\frac{(r+2)(r+1)^2n^{r+1}\alpha(n+r)!(n-r-3)!}{(n+\beta)^{r+2}n!(n-2)!}r!+\frac{(r+2)(r+1)\alpha^2n^r(n+r)!(n+r-2)!}{2(n+\beta)^{r+2}n!(n-2)!}r!\}+O(n^{-2}).\end{aligned}$$

The coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$, $f^{(r+2)}(x)$ in the above expression are respectively $r(r+2-\beta)$, $(1+r+\alpha)+x(2r+3-\beta)$ and $x(1+x)$. Taking the limits as $n\to\infty$ and by using induction hypothesis on r. To complete the proof of the theorem, it will be enough to show that $E_2\to 0$ as $n\to\infty$. To estimate E_2 using Lemma 5, we have

$$|E_2|\le(n-1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+1)^i\frac{|q_{i,j,r(x)}|}{x^r(1+x)^r}\sum_{k=0}^{\infty}b_{n,k}(x)|k-(n+1)x|^j\int_0^{\infty}p_{n,k}(t)|\varepsilon(t,x)|\left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2}dt.$$

For a given $\varepsilon>0$, there exist $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $|t-x|<\delta$. For $|t-x|\ge\delta$, since $\varepsilon(t,x)\to 0$ as $t\to x$, further if λ is any integer $\lambda>max(\gamma,r+2)$, then we find a constant $K>0$ such that $|\varepsilon(t,x)|\left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2}\le K\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\lambda}$. Thus

$$\begin{aligned}|E_2|\le\ &(n-1)A_1\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+1)^{i+1}\sum_{k=0}^{\infty}b_{n,k}(x)|k-(n+1)x|^j\{\int_{|t-x|<\delta}\varepsilon p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2}dt\\&+\int_{|t-x|\ge\delta}Kp_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\lambda}dt\}=:E_3+E_4.\end{aligned}$$

Now applying Schwarz inequality for the integration and summation, we have

$$\begin{aligned} |E_3| \leq & \quad (n-1)\varepsilon A_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^{i+1} \sum_{k=0}^{\infty} b_{n,k}(x)|[k-(n+1)x]|^j \\ & \times \left(\int_0^\infty p_{n,k}(t)dt\right)^{\frac{1}{2}} \left(\int_0^\infty p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2r+4} dt\right)^{\frac{1}{2}} \\ \leq & \quad (n-1)\varepsilon A_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^{i+1} \left(\sum_{k=0}^{\infty} b_{n,k}(x)|[k-(n+1)x]|^{2j}\right)^{\frac{1}{2}} \\ & \times \left(\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^\infty p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2r+4} dt\right)^{\frac{1}{2}}. \end{aligned}$$

Using Lemma 3 and Lemma 4, we get

$$|E_3| \leq (n-1)\varepsilon A_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^{i+1}.O(n^{j/2}).O(n^{-(r+2)/2}) \leq \varepsilon O(1).$$

For arbitrary chosen ε, we follow that $E_3 = o(1)$. Now using Schwarz inequality for the integration and summation, Lemma 3 and Lemma 4, we get

$$\begin{aligned} |E_4| \quad \leq & \quad (n-1)A_2 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^{i+1} \sum_{k=0}^{\infty} b_{n,k}(x)|k-(n+1)x|^j \int_{|t-x|\geq\delta} p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\lambda} dt \\ \leq & \quad (n-1)A_2 \sum_{\substack{2i+j\leq r \\ i,j\geq 0j}} (n+1)^{i+1} \left(\sum_{k=0}^{\infty} b_{n,k}(x)(k-(n+1)x)^{2j}\right)^{1/2} \\ & \times \left(\sum_{k=1}^{\infty} b_{n,k}(x)\int_o^\infty p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2\lambda} dt\right)^{1/2} \\ = & \quad (n-1) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^{i+1}.O(n^{j/2}).O(n^{-\lambda/2}) = O(n^{(r+2-\lambda)/2}) = o(1). \end{aligned}$$

Hence we get $E_2 \to 0$ as $n \to \infty$. Combining the estimates of E_1 and E_2, we get the required results. This completes the proof of theorem.

Theorem 2. Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $r \leq m \leq r+2$. If $f^{(m)}$ exist and is continuous on $(a-\eta, b+\eta) \subset [0,\infty)$, $\eta > 0$, then for n sufficiently large

$$\left\|B_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right\|_{c[a,b]} \leq A_1 n^{-1} \sum_{i=r}^{m} \left\|f^{(i)}\right\|_{c[a,b]} + A_2 n^{-1/2}\omega(f^{(m)}, n^{-1/2}) + O(n^{-2}),$$

where A_1 A_2 are constants which do not depend on f and n and $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$ and $\|.\|_{c[a,b]}$ being the sup-norm on $[a,b]$.

Proof: Using Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!}(t-x)^m \chi(t) + h(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function on the interval $(a-\eta, b+\eta)$. Now

$$\begin{aligned} B_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x) &= \left\{ \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} B_{n,\alpha,\beta}^{(r)}((t-x)^i) - f^{(r)}(x) \right\} \\ &\quad + B_{n,\alpha,\beta}^{(r)} \left(\frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!} (t-x)^m \chi(t), x \right) \\ &\quad + B_{n,\alpha,\beta}^{(r)}(h(t,x)(1-\chi(t)), x) =: J_1 + J_2 + J_3. \end{aligned}$$

By using Lemma 2, we have

$$\begin{aligned} J_1 &= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} [x^j \frac{n^j}{(n+\beta)^j} \frac{(n+j)!(n-j-2)!}{n!(n-2)!} \\ &\quad + x^{j-1} (j^2 \frac{n^j}{(n+\beta)^j} \frac{(n+j-1)!(n-j-2)!}{n!(n-2)!} \\ &\quad + j\alpha \frac{n^{j-1}}{(n+\beta)^j} \frac{(n+j-1)!(n-j-1)!}{n!(n-2)!}) \\ &\quad + x^{j-2} \{ j(j-1)^2 \alpha \frac{n^{j-1}}{(n+\beta)^j} \frac{(n+j-2)!(n-j-1)!}{n!(n-2)!} \\ &\quad + \frac{j(j-1)\alpha^2}{2} \frac{n^{j-2}}{(n+\beta)^j} \frac{(n+j-2)!(n-j)!}{n!(n-2)!} \} + O(n^{-2})] - f^{(r)}(x). \end{aligned}$$

Therefore, $\|J_1\|_{c|a,b|} \leq A_1 n^{-1} \sum_{i=r}^{m} \left\| f^{(i)} \right\|_{c|a,b|} + O(n^{-2})$, uniformly on $[a,b]$.
Next, we estimate J_2 as

$$\begin{aligned} \|J_2\| &\leq \left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} \left| b_{n,k}^r(x) \right| \int_0^\infty p_{n,k}(t) \left\{ \left| \frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!} \right| \left| \frac{nt+\alpha}{n+\beta} - x \right|^m \chi(t) \right\} dt \\ &\leq \frac{\omega(f^{(m)},\delta)}{m!} \left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} \left| b_{n,k}^r(x) \right| \int_0^\infty p_{n,k}(t) \left(1 + \frac{\left| \frac{nt+\alpha}{n+\beta} - x \right|^m}{\delta} \right) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt \\ &\leq \frac{\omega(f^{(m)},\delta)}{m!} \left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} \left| b_{n,k}^r(x) \right| \int_0^\infty p_{n,k}(t) \left(\left| \frac{nt+\alpha}{n+\beta} - x \right|^m + \delta^{-1} \left| \frac{nt+\alpha}{n+\beta} - x \right|^{m+1} \right) dt. \end{aligned}$$

Using Schwarz inequality for integration and summation, we get

$$\begin{aligned} & \left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} b_{n,k}(x) |k-(n+1)x|^j \int_0^\infty p_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^m dt \\ & \leq \left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} b_{n,k}(x) |k-(n+1)x|^j \left(\int_0^\infty p_{n,k}(t) dt \right)^{1/2} \left(\int_0^\infty p_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{2m} dt \right)^{1/2} \\ & \leq (n+1)^j \left(\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) |\frac{k}{n+1} - x|^{2j} \right)^{1/2} \left(\left(\frac{n-1}{n}\right) \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\infty p_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{2m} dt \right)^{1/2} \\ & = O(n^{j/2}).O(n^{-m/2}) = O(n^{(j-m)/2}), \end{aligned} \qquad (3.1)$$

uniformly on $[a, b]$.
Therefore by Lemma 5 and (3.1), we get

$$\left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}|b_{n,k}(x)|\int_0^{\infty}p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$
$$\leq \left(\frac{n-1}{n}\right)\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i|k-(n+1)x|^j\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}b_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$

$$\leq \left(\sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\right)\left(\frac{n-1}{n}\right)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i\left(\sum_{k=0}^{\infty}b_{n,k}(x)|k-(n+1)x|^j\int_0^{\infty}p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt\right)$$
$$= M\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i O(n^{(j-m)/2}) = O(n^{(r-m)/2}). \quad (3.2)$$

uniformly on $[a, b]$, where

$$M = \sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\sup_{x\in|a,b|}\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}.$$

Choosing $\delta = n^{-1/2}$ and applying (3.2), we obtain.

$$\|J_2\|_{c[a,b]} \leq \frac{\omega(f^{(m)}, n^{-1/2})}{m!}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i[O(n^{(j-m)/2}) + n^{-1/2}O(n^{(r-m-1)/2}] + O(n^{-m})$$
$$\leq M_2 n^{-(r-m)/2}\omega(f^{(m)}, n^{-1/2}).$$

For $t \in [0, \infty)\backslash(a-\eta, b+\eta)$, we can choose δ such that $|t-x| \geq \delta$ for all $x \in [a, b]$. Thus by Lemma 5, we get

$$|J_3| \leq M\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i b_{n,k}(x)|k-(n+1)x|^j\int_0^{\infty}p_{n,k}(t)|h(t,x)|.$$

For $|t-x| \geq \delta$, we can find a constant K such that $|h(t,x)| \leq K|\frac{nt+\alpha}{n+\beta} - x|^{\beta}$, where β is an integer $\geq max(\gamma, m)$. Hence applying the Schwarz inequality for both integration and summation, Lemma 3 and Lemma 4, it is easy to show that $J_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a, b]$. Combining the estimates of J_1, J_2, J_3, we get the required result.

4 Conclusions

The modification of operators plays an important role in approximation theory to obtain better approximation. In this paper, we extend the study of linear positive operators by applying hypergeometric series.

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Competing Interests

The authors declare that no competing interests exist.

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