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Sharp inequalities related to Wilker results

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Abstract: In this paper, we interested in Wilker inequalities. We provide finer bounds than known previous. Moreover, bounds are obtained for the following trigonometric function

$$g_n(x) = \left(\frac{\sin(x)}{x}\right)^2 \left(1 - \frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right) + \frac{\tan(x)}{x},\ n \geq 0.$$

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MSC: 26D07; 33B10; 33B20; 26D15.

1. Introduction

For $0 < x < \pi/2$, we know

$$\frac{\sin x}{x} < 1 < \frac{\tan x}{x},$$

or equivalently

$$\frac{x}{\tan x} < 1 < \frac{x}{\sin x}.$$

Exploiting these inequalities to the aim to provide other similar Wilker proposed open problems [1]. One of them is

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2.$$

Moreover, Wilker asked about the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x\,.$$

These problems was solved by Sumner *et al.*, [2] who proved in addition

$$2 + \frac{16}{\pi^4}x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x.$$

As we may remark these constants $\frac{16}{\pi^4}$ and $\frac{8}{45}$ are the limits at 0 and $\pi/2$ of the function

$$x \to \frac{(\frac{\sin x}{x})^2 + \frac{\tan x}{x} - 2}{x^3 \tan x}.$$

Chen and Cheung [3] proved that this function decreases and provided

$$2 + \frac{8x^3}{45} + \frac{16x^5}{315} \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8x^3}{45} + \left(\frac{2}{\pi}\right)^6 x^5 \tan x,$$
$$2 + \frac{8x^3}{45} + \frac{16x^5}{315} + \frac{104x^7}{4725} \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8x^3}{45} + \frac{16x^5}{315} + \left(\frac{12}{\pi}\right)^8 x^7 \tan x.$$

Moreover, all these constants are the best possible constants.

Later in using a natural approach Mortici [4] gave improvements of that inequalities and produced the following for $0 < x < \pi/2$;

$$2 + \frac{1}{\cos x}\left(\frac{8x^4}{45} - \frac{4x^6}{105}\right) < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x}\left(\frac{8x^4}{45}\right).$$

Maleševic *et al.*, [5] further refined the above inequality for $0 < x < \pi/2$ and $n \in IN$:

$$2 + \frac{1}{\cos x}\sum_{k=2}^{2n+1}(-1)^k D_k x^{2k} < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x}\sum_{k=2}^{2n}(-1)^k D_k x^{2k},$$

where $D_k = \frac{\left(-9+3^{2k+2}-40k-32k^2\right)}{4(2k+2)!}$.

On the other hand, Wu and Srivastava [[6], Lemma 3] proved the following dual inequality for $0 < x < \pi/2$;

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2, \qquad 0 < x < \frac{\pi}{2}.$$

Mortici [4] improved that result;

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \frac{2x^4}{45},$$

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > \frac{2\sin(x)}{x} + \frac{\tan(x)}{x}.$$

Maleševic *et al.*, [5] further improved the above result for $0 < x < \pi/2$ and $n \in IN$:

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \sum_{k=2}^{n}\frac{\mid B_{2k} \mid (2k-2)2^{2k}}{(2k)!}x^{2k}.$$

Theorem 1. *([5], Theorem 4 p.9) For $0 < x < \frac{\pi}{2}$;*

i)

$$2+\frac{1}{\cos x}\left(\frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} - \frac{37x^{10}}{133650}\right) < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x}\left(\frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725}\right).$$

ii)

$$2+\frac{1}{\cos x}\left(\frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} - \frac{37x^{10}}{133650} + \frac{283x^{12}}{20638800} - \frac{3503x^{14}}{6810604000}\right)$$
$$< \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x}\left(\frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} - \frac{37x^{10}}{133650} + \frac{283x^{12}}{20638800}\right).$$

Theorem 2. *([5], Theorem 5 p.11) For $0 < x < \frac{\pi}{2}$;*

i)

$$2 + \frac{2x^4}{45} + \frac{8x^6}{945} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < 2 + \frac{2x^4}{45} + (\frac{2}{\pi})^6\left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360}\right)x^6.$$

ii)

$$2+\frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)}$$
$$< 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + (\frac{2}{\pi})^8\left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560}\right)x^8.$$

iii)

$$2+\frac{2x^4}{45}+\frac{8x^6}{945}+\frac{2x^8}{1575}+\frac{16x^{10}}{93555}<\left(\frac{x}{\sin(x)}\right)^2+\frac{x}{\tan(x)}$$
$$<2+\frac{2x^4}{45}+\frac{8x^6}{945}+\frac{2x^8}{1575}+(\frac{2}{\pi})^{10}\left(-2+\frac{\pi^2}{4}-\frac{\pi^4}{360}-\frac{\pi^6}{7560}-\frac{\pi^8}{201600}\right)x^{10}.$$

In this paper, we aim to refine the inequalities from Theorems 1 and 2.

2. The first Wilker inequality

The first Wilker inequality [1];

$$\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}>2, \qquad x\in(0,\frac{\pi}{2})$$

was intensively studied by many authors, see for example, [7–14].

Mortici [4] and Maleševic *et al.*, [5] proved

$$2+\left(\frac{8}{45}x^4-\frac{8}{105}x^6\right)\left(\frac{1}{\cos x}\right)<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}<2+\left(\frac{8x^4}{45\cos(x)}\right).$$

The following result provides bounds permitting to refine some previous ones;

Theorem 3. *For $0<x<\pi/2$, the following inequalities holds for $1\le m\le n$ and $p\le n$;*

$$\left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2^{\beta})}x^{2k}-\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{n}a^kx^{2k-2}\right)$$
$$\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}\right)$$
$$<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}$$
$$<\left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}+\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{\infty}a^kx^{2k-2}\right)$$
$$\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}\right),$$

where B_{2k} are the Bernoulli numbers and $\beta=2+\frac{ln(1-\frac{6}{\pi^2})}{ln2}=0.6491...$ is the Alzer constant.

Proof. First we may write obviously

$$\left(\frac{\sin(x)}{x}\right)^2\left(1+\frac{2x}{\sin(2x)}\right)=\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

Lemma 1. *For $0<x<\pi2$, the following inequalities holds for any integer $m\ge5$;*

$$\left(\frac{\sin(x)}{x}\right)^2<1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}+\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{\infty}a^kx^{2k-2},$$

$$1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2)}x^{2k}-\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{n}a^kx^{2k-2}<\left(\frac{\sin(x)}{x}\right)^2.$$

Lemma 2. *For $0<x<\pi/2$, consider the function*

$$f(x)=(\sin x)^2-x^3\cot x-\frac{x^6}{15}+\frac{x^8}{945},$$

then $f(x)$ can be expressed as power series

$$f(x) = \sum_{k\geq 5} a^k x^{2k},$$

with

$$a_k = \frac{2^{2k-2}\,|B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!} = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)\,k}\right),$$

where B_{2k} are the Bernoulli numbers. Moreover, the coefficients $a_k, k \geq 5$ are all positive:

$$a_5 = \frac{1}{255150},\ a_6 = \frac{2}{15436575},\ a_7 = \frac{103}{8300667375},\ldots.$$

Proof. The following series expansions can be found in [15–17]

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}\mid B_{2k}\mid}{(2k)!} x^{2k-1}, \qquad x \in (0,\pi),$$

and

$$\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k}, \qquad x \in \left(0, \frac{\pi}{2}\right).$$

$$(\sin x)^2 - x^3 \cot x = \left(\frac{1}{15}x^6 - \frac{1}{945}x^8 + \frac{1}{2835}x^{10} + \frac{8}{467775}x^{12} + \frac{206}{91216125}x^{14} + \frac{139}{638512875}x^{16} + \frac{10861}{488462349375}x^{18} + \frac{438628}{194896477400625}x^{20} + O\left(x^{22}\right)\right).$$

To prove the positivity of the coefficients a_k, k even we will use the following inequality for Bernoulli numbers established by D'aniello [18]:

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} < \mid B_{2k} \mid < \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}.$$

For any odd value of k, we have

$$a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)\,k}\right) = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{1}{(2k-1)\,k}\right) > 0.$$

Consider the even case, then

$$a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{-1}{(2k-1)\,k}\right).$$

By definition of Bernoulli numbers

$$S_n(p) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k} B_k n^{p+1-k}.$$

Then for k even

$$a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| - \frac{1}{(2k-1)\,k}\right) > \frac{2^{2k-2}}{(2k-2)!}\left(\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)} - \frac{1}{(2k-1)\,k}\right),$$

and

$$a_k < \frac{2^{2k-2}}{(2k-2)!}\left(\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)\,k}\right).$$

Therefore

$$\frac{2^{2k-2}\frac{2(2k-2)!}{\pi^{2k-2}\left(2^{2k-2}-1\right)}}{(2k-2)!}+\frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!}<a_k=\frac{2^{2k-2}\,|B_{2k-2}|}{(2k-2)!}+\frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!}$$
$$<\frac{2^{2k-2}\frac{2(2k-2)!}{\pi^{2k-2}\left(2^{2k-2}-2\right)}}{(2k-2)!}+\frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!},$$

and

$$\sum_{k\geq 5}\left(\frac{2^{2k-1}}{2\pi^{2k-2}\left(2^{2k-2}-1\right)}+\frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!}\right)x^{2k}<f(x).$$

Thus it implies the left hand, since for any integer $n\geq 5$, we have

$$\begin{aligned}\cos x&\leq\cos x+\sin x\left(\frac{x^3}{15}-\frac{x^5}{945}\right)+\sin x\sum_{5\leq k\leq n}a_kx^{2k-3}\\&\leq\cos x+\sin x\left(\frac{x^3}{15}-\frac{x^5}{945}\right)+\sin x\sum_{5\leq k\leq\infty}a_kx^{2k-3}\\&=\left(\frac{\sin x}{x}\right)^3.\end{aligned}$$

$$\begin{aligned}(\sin x)^2-x^3\cot x&=(\sin x)^2-x^3\left(\frac{1}{x}-\sum_{k=1}^{\infty}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k-1}\right)\\&=(\sin x)^2-x^2+\sum_{k=1}^{\infty}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k+2}\\&<(\sin x)^2-x^2+\sum_{k=1}^{\infty}\frac{2^{2k+1}x^{2k+2}}{\pi^{2k}\left(2^{2k}-2\right)}.\end{aligned}$$

In the other hand we know that for $k>1$;

$$(2k)!>\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}e^{\frac{1}{24k+1}}.$$

It implies

$$\begin{aligned}\left(\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)}-\frac{1}{(2k-1)\,k}\right)&>\left(\frac{2\sqrt{4\pi(k-1)}(\frac{2k-2}{e})^{2k-2}e^{\frac{1}{48k-23}}}{\pi^{2k-2}(2^{2k-2}-1)}-\frac{1}{(2k-1)\,k}\right)\\&>\left(\frac{2\sqrt{4\pi(k-1)}(\frac{2k-2}{\pi e})^{2k-2}}{(2^{2k-2}-1)}-\frac{1}{(2k-1)\,k}\right)\\&>\left(\frac{2\sqrt{4\pi(k-1)}(\frac{2k-2}{\pi e})^{2k-2}}{2^{2k-2}}-\frac{1}{(2k-1)\,k}\right)\\&>\left(2\sqrt{4\pi(k-1)}(\frac{k-1}{\pi e})^{2k-2}-\frac{1}{(2k-1)\,k}\right).\end{aligned}$$

Thanks to *Maple*, we may easily verify that the last expression is non negative as soon as $k>6$. This means that a_k are non negative. □

By this Lemma, we deduce

$$\frac{f(x)}{x^2}=\left(\frac{\sin(x)}{x}\right)^2-x\cot x-\frac{x^4}{15}+\frac{x^6}{945}=\sum_{k\geq 5}a^kx^{2k-2},$$

where
$$a_k = \frac{2^{2\,k-2}}{(2\,k-2)!}\left(|B_{2\,k-2}| + \frac{(-1)^{k+1}}{(2\,k-1)\,k}\right).$$
Thus for any integer $n \geq 5$
$$\left(\frac{\sin(x)}{x}\right)^2 = x\cot x + \frac{x^4}{15} - \frac{x^6}{945} + \sum_{k\geq 5} a^k x^{2k-2} \geq x\cot x + \frac{x^4}{15} - \frac{x^6}{945} + \sum_{k=5}^{n} a^k x^{2k-2},$$

since coefficients $a_k > 0$.

Therefore, since by [[17], p.145], for $x \in (0, \pi)$,
$$\begin{aligned}\cot x &= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...\\ &= \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1}\\ &= \frac{1}{x} - \sum_{k=1}^{n-1} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1} - \sum_{k=n}^{\infty} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1},\end{aligned}$$

and by [18]
$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} <\mid B_{2k} \mid< \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}. \tag{1}$$
Then we deduce the inequalities
$$\sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k-1} < \cot x - \frac{1}{x} + \sum_{k=1}^{n} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1} < \sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k-1}.$$

Notice that Alzer [19] provides a further result,
$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} <\mid B_{2k} \mid< \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2^{\beta})} < \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2)}, \tag{2}$$

where $\beta = 2 + \frac{ln(1-\frac{6}{\pi^2})}{ln2} = 0.6491...$ is the best possible constant in the sense that it can not be replaced respectively by any bigger and smaller constant in the double inequality.

We then obtain better inequalities for $\cot x$,
$$\sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-2^{\beta})} x^{2k-1} < \cot x - \frac{1}{x} + \sum_{k=1}^{n} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1}.$$

On the other hand, it follows by the same way for any integer $m \geq 5$,
$$\left(\frac{\sin(x)}{x}\right)^2 < 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} - \sum_{k=1}^{m} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} + \sum_{k=3}^{\infty} a^k x^{2k-2},$$

implies
$$1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2^{\beta})} x^{2k} - \sum_{k=1}^{m} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} + \sum_{k=3}^{n} a^k x^{2k-2} < \left(\frac{\sin(x)}{x}\right)^2.$$
Let us consider now expansions trigonometric functions with power series. We will use the Taylor expansions of $\sin(x)$,
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin\theta x}{(2k+1)!} x^{2k+1},$$

where $0 < \theta < 1$.

It is easy to remark that
$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!},$$

for $0 < x < \frac{\pi}{2}$. We then deduce bounds for $\left(\frac{\sin x}{x}\right)^2$,

$$\sum_{k=1}^{2p}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2} < \left(\frac{\sin x}{x}\right)^2 < \sum_{k=1}^{2p+1}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}.$$

On the other hand, we know that [[17] p.145]

$$\frac{1}{\sin x} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + ... = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k-1}.$$

We then derive the following inequalities for any $n \geq 1$.

Lemma 3. *For $0 < x < \pi/2$, the following inequalities holds for any integer $p \geq 1$,*

$$\begin{aligned} 2+&\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} + \sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k} \\ &< 1 + \frac{2x}{\sin 2x} \\ &< 2 + \sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} + 2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}. \end{aligned}$$

Indeed, by [[17] p.146] one has

$$1 + \frac{2x}{\sin 2x} = 2 + \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} > 2 + \sum_{k=1}^{n}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k}.$$

Then it follows by (1)

$$\begin{aligned} 2+&\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} + \sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k} \\ &< 1 + \frac{2x}{\sin 2x} \\ &< 2 + \sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} + 2\sum_{k=p+1}^{\infty}\frac{2^{2k}}{\pi^{2k}}x^{2k} \\ &= 2 + \sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} + 2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}. \end{aligned}$$

We may also prove the following frame

$$\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2} - 2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k} < 1 + \frac{2x}{\sin 2x} < \frac{2}{1-(\frac{2x}{\pi})^2}.$$

Finally, we get a lower bound for the product

$$\begin{aligned} \left(\frac{\sin(x)}{x}\right)^2\left(1+\frac{2x}{\sin 2x}\right) &> \left(1 - \sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2^{\beta})}x^{2k} - \sum_{k=1}^{m}\frac{2^{2k} \mid B_{2k} \mid}{(2k)!}x^{2k} + \sum_{k=3}^{n}a^k x^{2k-2}\right) \\ &\times\left(2 + \sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!}x^{2k} + \sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}\right) \\ &= \phi_{m,n,p}(x). \end{aligned}$$

The upper bound is

$$\left(\frac{\sin(x)}{x}\right)^2\left(1+\frac{2x}{\sin 2x}\right) < \left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}-\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{\infty}a^k x^{2k-2}\right)$$
$$\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}\right)$$
$$=\psi_{m,p}(x).$$

Thus the following inequalities hold for $0<x<\pi/2$ and for integers $1\leq m;\ 3\leq n;\ 1\leq p\leq n$,

$$\phi_{m,n,p}(x) < \left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x} < \psi_{m,p}(x).$$

Theorem 3 is then proved. □

Let $0<x<\pi/2$. By Theorem 3 we are able to precise the lower bound of the Wilker inequality in putting different values of n, p.

Example 1. Taking $n=3$ and $p=m=2$, we have

$$\phi_{2,4,2}=2+\frac{8x^4}{45} < \left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

Then we find again a result of [7] since

$$\frac{1}{\cos x}\left(\frac{8x^4}{45}-\frac{4x^6}{105}\right) > \frac{\frac{8x^4}{45}-\frac{4x^6}{105}}{1-\frac{x^2}{2!}+\frac{x^4}{4!}} > \frac{8x^4}{45}.$$

Example 2. Taking $n=4$ and $p=m=3$, we find

$$\phi_{3,4,3}=2+\frac{8x^4}{45}+\frac{16x^6}{315} < \left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

Example 3. Taking $n=5$ and $p=m=3$, we have

$$\phi_{3,5,3}=2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725} < \left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

Example 4. Taking $n=6$ and $p=m=5$, we have

$$\phi_{5,6,5}=2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825} < \left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

This permits to find again Theorem 1(i) [[8], p.9]. Indeed, since (see Lemma 4 below) for $0<x<\pi/2$,

$$\cos x > 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!}-\frac{x^{14}}{14!},$$

we then deduce

$$(\cos x)\left(\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}\right) > \frac{8}{45}x^4-\frac{4}{105}x^6+\frac{19}{4725}x^8-\frac{37}{133650}x^{10}.$$

Example 5. Taking $n=7$ and $p=m=4$, we obtain

$$\phi_{4,7,4}=2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}+\frac{152912x^{12}}{42567525} < \left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

By the same way, using again the lower bound of $\cos x$, we find a result of [[8], p.10],

$$(\cos x)\left(2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}+\frac{152912x^{12}}{42567525}\right)$$
$$>\frac{8}{45}x^4-\frac{4}{105}x^6+\frac{19}{4725}x^8-\frac{37}{133650}x^{10}+\frac{283}{20638800}x^{12}-\frac{3503}{6810804000}x^{14}.$$

In the sequel we will find upper and lower bounds of Wilker inequalities which appear to be finer than known previous. Consider at first;

Lemma 4. *For $0<x<\pi/2$, the following inequalities holds for any integer $p\geq 1$,*

$$\sum_{k=1}^{2p}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}<\left(\frac{\sin x}{x}\right)^2<\sum_{k=1}^{2p+1}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2},$$
$$\sum_{k=1}^{2p+1}(-1)^k\frac{1}{(2k)!}x^{2k}<\cos x<\sum_{k=1}^{2p}(-1)^k\frac{1}{(2k)!}x^{2k}.$$

Let us consider expansions trigonometric functions with power series. We will use the Taylor expansions of $\sin x,\ \cos x$,

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+....+(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}+(-1)^k\frac{\sin\theta x}{(2k+1)!}x^{2k+1},$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+....+(-1)^k\frac{x^{2k}}{(2k)!}+(-1)^{k+1}\frac{\cos\theta x}{(2k+2)!}x^{2k+2},$$

where $0<\theta<1$.

It is easy to remark that

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}<\frac{\sin x}{x}<1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!},$$

for $0<x<\frac{\pi}{2}$. We then deduce bounds for $\left(\frac{\sin x}{x}\right)^2\cos x$,

$$\sum_{k=1}^{2p}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}<\left(\frac{\sin x}{x}\right)^2<\sum_{k=1}^{2p+1}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}.$$
$$\sum_{k=1}^{2p+1}(-1)^k\frac{1}{(2k)!}x^{2k}<\cos x<\sum_{k=1}^{2p}(-1)^k\frac{1}{(2k)!}x^{2k}.$$

By Lemma 3, we then derive the following which improves Theorem 3.

Theorem 4. *For $0<x<\pi/2$ the following inequalities holds for any $q\geq 1, 1\leq p\leq n$,*

$$\left(\sum_{k=1}^{2q}(-1)^{k+1}\frac{2^{2k-1}}{(2k)!}x^{2k-2}\right)\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}\right)$$
$$<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}$$
$$<\left(\sum_{k=1}^{2q+1}(-1)^{k+1}\frac{2^{2k-1}}{(2k)!}x^{2k-2}\right)\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}\right),$$

where B_{2k} are the Bernoulli numbers.

Corollary 5. *For $0 < x < \pi/2$ the following inequalities holds for any $n, p, \quad 1 \le p \le n$,*

$$\left(\frac{\sin(x)}{x}\right)^2\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}-2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k}\right)$$
$$<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}$$
$$<\left(\frac{\sin(x)}{x}\right)^2\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right),$$

where B_{2k} are the Bernoulli numbers.

Corollary 5 means that the following inequalities hold:

$$\left(\frac{\sin(x)}{x}\right)^2\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}-2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k}\right)$$
$$<g_n(x)$$
$$=\left(\frac{\sin(x)}{x}\right)^2\left(1-\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right)+\frac{\tan(x)}{x}$$
$$<\left(\frac{\sin(x)}{x}\right)^2\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}\right)$$
$$<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

The function $g_n(x)$ is growing as n increasing. We have

$$g_2(x)=\left(\frac{\sin(x)}{x}\right)^2\left(1-\frac{2\left(\frac{2x}{\pi}\right)^2}{1-(\frac{2x}{\pi})^2}\right)+\frac{\tan(x)}{x}<g_n(x)<lim_{n\to\infty}g_n(x)=\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}.$$

Moreover, we may compute the limit when x tends to $(\frac{\pi}{2})^-$,

$$lim_{x\to\frac{\pi}{2}}g_n(x)=\frac{2(3+4n)}{\pi^2},$$

and

$$g_n(x)-\frac{2(3+4n)}{\pi^2}=\frac{-24\,n^2+24\,n+33+2\,\pi^2}{3\pi^3}(\pi-2x)+\dots.$$

Let $0 < x < \pi/2$. Then we have following examples;

Example 6. Taking $n = 3$ and $p = 2$, we find

$$2+\frac{8x^4}{45}<g_3(x)=\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-\left(\frac{\sin x}{x}\right)^2\left(\frac{2\left(\frac{2x}{\pi}\right)^8}{1-(\frac{2x}{\pi})^2}\right)$$
$$<\frac{2306x^{10}}{467775}-\frac{2x^8}{63}+\frac{16x^6}{315}+\frac{8x^4}{45}+2<\frac{16x^6}{315}+\frac{8x^4}{45}+2.$$

Example 7. Taking $n = 4$ and $p = 3$, we find

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}<g_4(x)=\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}-\left(\frac{\sin(x)}{x}\right)^2\left(\frac{2\left(\frac{2x}{\pi}\right)^{10}}{1-(\frac{2x}{\pi})^2}\right)$$
$$<\frac{61232x^{12}}{30405375}-\frac{868x^{10}}{66825}+\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2<\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2.$$

Example 8. Taking $n = 5$ and $p = 3$, we obtain

$$2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} < g_5(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - \left(\frac{\sin(x)}{x}\right)^2 \left(\frac{2\left(\frac{2x}{\pi}\right)^{12}}{1-(\frac{2x}{\pi})^2}\right)$$
$$< \frac{1566172x^{14}}{1915538625} - \frac{480604x^{12}}{91216125} + \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2$$
$$< \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2.$$

We then improve by another way Theorem 1(i) ([[8], p.9]), since by Lemma 3,

$$\left(\frac{8}{45}x^4 + \frac{16}{315}x^6 + \frac{104}{4725}x^8 + \frac{592}{66825}x^{10}\right)\cos(x) < \frac{8}{45}x^4 - \frac{4}{105}x^6 + \frac{19}{4725}x^8$$
$$\left(\frac{8}{45}x^4 + \frac{16}{315}x^6 + \frac{104}{4725}x^8 + \frac{592}{66825}x^{10} + \frac{152912}{42567525}x^{12}\right)\cos(x)$$
$$> \frac{8}{45}x^4 - \frac{4}{105}x^6 + \frac{19}{4725}x^8 - \frac{37}{133650}x^{10} + \frac{283}{20638800}x^{12}.$$

Example 9. Taking $n = 6$ and $p = 3$, we obtain

$$2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} + \frac{592}{66825}x^{10} < g_6(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - \left(\frac{\sin(x)}{x}\right)^2 \left(\frac{2\left(\frac{2x}{\pi}\right)^{14}}{1-(\frac{2x}{\pi})^2}\right)$$
$$< \frac{161934166x^{16}}{488462349375} - \frac{123992x^{14}}{58046625} + \frac{152912x^{12}}{42567525} + \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2$$
$$< \frac{152912x^{12}}{42567525} + \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2.$$

The last estimate improves Theorem 1(ii) ([[8], p.10]).

3. Second Wilker inequality

Wu and Srivastava [[14], Lemma 3] proved the following dual inequality for $0 < x < \pi/2$ also called second Wilker inequality

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2, \qquad 0 < x < \frac{\pi}{2}.$$

Mortici [4] improved it and gave,

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \frac{2x^4}{45}.$$
$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > \frac{2\sin(x)}{x} + \frac{\tan(x)}{x}.$$

Maleševic *et al.*, [[5], Theorem 5] gave an improvement of that and provided the following bounds for $0 < x < \pi/2$ and any integer $m \geq 2$

$$2 + \sum_{k=2}^{m} \frac{(2k-2)2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)}$$
$$< 2 + \sum_{k=2}^{m-1} \frac{(2k-2)2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} + \left(\frac{2x}{\pi}\right)^{2n} \left(\frac{\pi^2}{4} - 2 - 2 + \sum_{k=2}^{m-1} \frac{(2k-2)2^{2k} \mid B_{2k} \mid}{(2k)!} \left(\frac{\pi}{2}\right)^{2k}\right).$$

It seems that Theorem 6 below improves that result.

Theorem 6. *For $0 < x < \pi/2$ and any $p \geq 1$, the following inequalities hold:*

$$\left(1+\sum_{k=1}^{2p}(-1)^k\frac{(2x)^{2k}}{(2k+1)!}\right)\left(1+\sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{\infty}\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}\right)$$
$$<\left(\frac{x}{\sin(x)}\right)^2+\frac{x}{\tan(x)}$$
$$<\left(1+\sum_{k=1}^{2p+1}(-1)^k\frac{(2x)^{2k}}{(2k+1)!}\right)\left(1+\sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{\infty}\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2^{\beta})}x^{2k}\right),$$

where B_{2k} are the Bernoulli numbers and $\beta = 2+\frac{ln(1-\frac{6}{\pi^2})}{ln2} = 0.6491...$ is the Alzer constant.

Proof. Recall the following series expansions which can be found in [17] or [16].

$$\cot x = \frac{1}{x}-\sum_{k=1}^{\infty}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k-1}, \quad \frac{\sin x}{x}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}.$$

Then we derive

$$\left(\frac{x}{\sin x}\right)^2 = -x^2\frac{d\cot x}{dx} = 1+\sum_{k=1}^{\infty}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}.$$

Notice that

$$\left(\frac{x}{\sin(x)}\right)^2+\frac{x}{\tan(x)} = \left(\frac{x}{\sin x}\right)^2\left(1+\frac{\sin 2x}{2x}\right)$$
$$= \left(1+\sum_{k=1}^{\infty}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}\right)\left(1+\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}x^{2k}}{(2k+1)!}\right).$$

By Taylor expansions, we may deduce

$$\sum_{k=0}^{2p}(-1)^k\frac{x^{2k}}{(2k+1)!} < \frac{\sin x}{x} < \sum_{k=0}^{2p+1}(-1)^k\frac{x^{2k}}{(2k+1)!}.$$

On the other hand, writing

$$\sum_{k=1}^{\infty}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k} = \sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{\infty}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}.$$

Combining with inequalities (2),

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} <\mid B_{2k}\mid< \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2^{\beta})} < \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2)}.$$

It follows for $0 < x < \pi/2$ and any $p \geq 1$,

$$\left(\frac{x}{\sin(x)}\right)^2+\frac{x}{\tan(x)} > \left(1+\sum_{k=1}^{2p}(-1)^k\frac{(2x)^{2k}}{(2k+1)!}\right)\left(1+\sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{\infty}\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}\right),$$

as well as

$$\left(\frac{x}{\sin(x)}\right)^2+\frac{x}{\tan(x)} < \left(1+\sum_{k=1}^{2p+1}(-1)^k\frac{(2x)^{2k}}{(2k+1)!}\right)\left(1+\sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{\infty}\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2^{\beta})}x^{2k}\right)$$
$$< \left(1+\sum_{k=1}^{2p+1}(-1)^k\frac{(2x)^{2k}}{(2k+1)!}\right)\left(1+\sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{\infty}\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)}x^{2k}\right).$$

□

Corollary 7. *For* $0 < x < \pi/2$ *and any* $n \geq p \geq 1$, *the following inequalities hold:*

$$a_{p,n}(x) = \left(1 + \sum_{k=1}^{2p} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right)\left(1 + \sum_{k=1}^{p} \frac{(2k-1)2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} + \sum_{k=p+1}^{n} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k}\right)$$
$$< \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < b_p(x) = \left(1 + \sum_{k=1}^{2p+1} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right)\left(1 + \sum_{k=1}^{p} \frac{(2k-1)2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k}\right.$$
$$\left. + \frac{2\left(\frac{x}{\pi}\right)^{2p+2}}{(1-(\frac{x}{\pi})^2)^2}\left[(1-2p)\left(\frac{x}{\pi}\right)^2 + 2p + 1\right] + \frac{2\left(\frac{x}{2\pi}\right)^{2p+2}}{(1-(\frac{x}{2\pi})^2)^2}\left[(1-2p)\left(\frac{x}{2\pi}\right)^2 + 2p + 1\right]\right).$$

Proof. Let us consider the function

$$\frac{1}{1-2^{1-2x}}, \quad x \geq 1.$$

For $1 \leq x$ this function can be majored as;

Lemma 5. *For* $1 \leq x$, *the following inequality holds:*

$$\frac{1}{1-2^{1-2x}} \leq (1 + 2^{2-2x}).$$

Indeed, consider the difference

$$\alpha(x) = \frac{1}{1-2^{1-2x}} < (1 + 2^{2-2x}).$$

Its derivative is

$$\frac{d\alpha(x)}{dx} = -2\,\frac{2^{1-2x}\ln(2)}{(1-2^{1-2x})^2} + 4\,2^{1-2x}\ln(2) = 2\,\frac{2^{1-2x}\ln(2)\left(1 - 4.\,2^{1-2x} + 2\,\left(2^{1-2x}\right)^2\right)}{(-1+2^{1-2x})^2}.$$

The study suggest us that this function is non positive for $x \neq 1$. It is such that $\alpha(1) = 0$, $\lim_{x\to\infty} \alpha(x) = 0$ and increases between 1 and ∞. Thus, we get for $1 \leq x$, $\frac{1}{1-2^{1-2x}} \leq (1 + 4.2^{-2x})$.

By Lemma 5 we have for any $k \geq 1$;

$$\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} = 2(2k-1)\frac{1}{1-2^{1-2x}}\left(\frac{x}{\pi}\right)^{2k}$$
$$\leq 2(2k-1)(1 + 4.2^{-2k})\left(\frac{x}{\pi}\right)^{2k}$$
$$= 2(2k-1)\left(\frac{x}{\pi}\right)^{2k} + 8(2k-1)\left(\frac{x}{2\pi}\right)^{2k}.$$

We then deduce the sum, since $0 < x < \pi/2$,

$$\sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} \leq \sum_{k=p+1}^{\infty} 2(2k-1)\left(\frac{x}{\pi}\right)^{2k} + 8(2k-1)\left(\frac{x}{2\pi}\right)^{2k}$$
$$= \frac{2\left(\frac{x}{\pi}\right)^{2p+2}}{(1-(\frac{x}{\pi})^2)^2}\left[(1-2p)\left(\frac{x}{\pi}\right)^2 + 2p + 1\right] + \frac{2\left(\frac{x}{2\pi}\right)^{2p+2}}{(1-(\frac{x}{2\pi})^2)^2}\left[(1-2p)\left(\frac{x}{2\pi}\right)^2 + 2p + 1\right].$$

Replacing in Theorem 6, we get Corollary 7. □

Let $0 < x < \pi/2$, we have following examples. We will use *Maple*.

Example 10. Taking $n = 3$ and $p = 2$, one obtains

$$\begin{aligned} a_{2,3}(x) &= 2 + \frac{2x^4}{45} + \frac{8x^6}{945} \\ &< \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} \\ &< b_2(x) \\ &= \left(2 - \frac{2x^2}{3} + \frac{2x^4}{15} - \frac{4x^6}{315}\right)\left(1 + \frac{x^2}{3} + \frac{x^4}{15} + \frac{85}{8}\frac{x^6}{\pi^6}\right) \\ &< 2 + \frac{2x^4}{45} + (\frac{85}{4\pi^6} - \frac{4}{315})x^6 . \end{aligned}$$

The last expression is lower than

$$2 + \frac{2x^4}{45} + \frac{64}{\pi^6}\left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360}\right)x^6 ,$$

since $(\frac{85}{4\pi^6} - \frac{4}{315}) - \frac{64}{\pi^6}(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360}) \approx -0.003697$. Then, this case is finer than Theorem 2(i).

Example 11. Taking $n = 4$ and $p = 3$, one obtains

$$\begin{aligned} a_{3,4}(x) =& 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} \\ <& \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} \\ <& b_3(x) \\ =& \left(2 - \frac{2x^2}{3} + \frac{2x^4}{15} - \frac{4x^6}{315}\right)\left(1 + \frac{x^2}{3} + \frac{x^4}{15} + \frac{2}{189}x^6 + \frac{16\,x^8\pi^2 - 12\,x^{10}}{\pi^6\,(x^2 - \pi^2)^2}\right. \\ &\left. + \frac{1}{4}\frac{16\,x^8\pi^2 - 3\,x^{10}}{\pi^6\,(x^2 - 4\,\pi^2)^2} - 2\,\frac{x^8}{\pi^6\,(-x^2 + \pi^2)} + \frac{x^8}{-32\,\pi^8 + 8\,x^2\pi^6}\right) \\ <& 2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \left(-\frac{34}{14175} + \frac{455}{16}\pi^{-8}\right)x^8 . \end{aligned}$$

The last expression is lower than

$$2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \frac{256}{\pi^8}\left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560}\right)x^8 ,$$

since $-\frac{34}{14175} + \frac{455}{16}\pi^{-8} - \frac{256}{\pi^8}\left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560}\right) \approx -0.0012804$. This case is also finer than Theorem 2(ii).

Example 12. Taking $n = 5$ and $p = 4$, one obtains (details omitted)

$$\begin{aligned} a_{3,4}(x) &= 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} + \frac{16x^{10}}{93555} \\ &< \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} \\ &< b_3(x) \\ &< 2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \frac{2}{1575}x^8 + \left(-\frac{4}{18711} + \frac{2313}{64}\pi^{-10}\right)x^{10} . \end{aligned}$$

The last expression is lower than

$$2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} + (\frac{2}{\pi})^{10}\left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560} - \frac{\pi^8}{201600}\right)x^{10} ,$$

since $\left(-\frac{4}{18711}+\frac{2313}{64}\right)-\left(-2+\frac{\pi^2}{4}-\frac{\pi^4}{360}-\frac{\pi^6}{7560}-\frac{\pi^8}{201600}\right)\approx -0.00007469$. This case is also finer than Theorem 2(iii).

4. Concluding remarks

1. Examples 1-5 and Examples 6-9 above suggest us that Corollary 7 is finer than Theorem 4 of [5]. Therefore, we may ask if the following is valid for $0<x<\pi/2$ and $n\geq 2$:
(i)
$$2+\frac{1}{\cos x}\sum_{k=2}^{2n+1}(-1)^k D_k x^{2k} < g_2(x)=\left(\frac{\sin(x)}{x}\right)^2\left(1-\frac{2\left(\frac{2x}{\pi}\right)^6}{1-(\frac{2x}{\pi})^2}\right)+\frac{\tan(x)}{x}.$$
Recall that for $n\geq 2$,
$$g_2(x)<g_n(x)=\left(\frac{\sin(x)}{x}\right)^2\left(1-\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right)+\frac{\tan(x)}{x}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$
(ii)
$$2+\frac{1}{\cos x}\sum_{k=2}^{2n+1}(-1)^k D_k x^{2k}<\left(\frac{\sin(x)}{x}\right)^2\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}-2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k}\right),$$
and
(iii)
$$\left(\frac{\sin(x)}{x}\right)^2\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right)<2+\frac{1}{\cos x}\sum_{k=2}^{2n}(-1)^k D_k x^{2k},$$
where $D_k=\frac{\left(-9+3^{2k+2}-40k-32k^2\right)}{4(2k+2)!}$.

2. Examples 10-12 suggest us that Corollary 7 is finer than Theorem 5 of [5] and we may naturally ask if more generally the following inequality
$$b_m(x)<2+\sum_{k=2}^{m-1}\frac{\mid B_{2k}\mid(2k-2)4^k}{(2k)!}x^{2k}+\left(\frac{2x}{\pi}\right)^{2n}\left(\frac{\pi^2}{4}-2-\sum_{k=2}^{m-1}\frac{\mid B_{2k}\mid(2k-2)4^k}{(2k)!}(\frac{\pi}{2})^{2k}\right),$$
holds for any $m\geq 2$, where
$$b_p(x)=\left(1+\sum_{k=1}^{2p+1}(-1)^k\frac{(2x)^{2k}}{(2k+1)!}\right)\left(1+\sum_{k=1}^{p}\frac{(2k-1)2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}\right.$$
$$\left.+\frac{2\left(\frac{x}{\pi}\right)^{2p+2}}{(1-(\frac{x}{\pi})^2)^2}\left[(1-2p)\left(\frac{x}{\pi}\right)^2+2p+1\right]+\frac{2\left(\frac{x}{2\pi}\right)^{2p+2}}{(1-(\frac{x}{2\pi})^2)^2}\left[(1-2p)\left(\frac{x}{2\pi}\right)^2+2p+1\right]\right).$$

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